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Applying the fuzzy homotopy analysis method to solve fuzzy initial value problems with variable coefficients

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ABSTRACT

In this study, we applied the fuzzy homotopy analysis method to find an approximate but analytical solution to initial value problems that have fuzzy variable coefficients. These coefficients were represented by triangular fuzzy functions. This approach lets us solve the fuzzy differential equations as an endless series of fuzzy numbers, where each fuzzy component can be determined without much difficulty. Based on the numerical data we analyzed, the series solutions we arrived at are both precise and closely match the exact analytical solutions for fuzzy problems.

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1. Introduction

Various methods have been developed to solve fuzzy differential equations (FDEs) with either fuzzy initial or boundary conditions. FDEs are significant in multiple science and engineering fields, leading to the development of exact-analytical, approximate analytical, and numerical solutions. Often, finding an exact analytical solution is challenging or impossible, which necessitates using numerical or approximate analytical solutions instead.

In terms of FDEs, the coefficients can either be non-fuzzy variable coefficients or fuzzy variable coefficients. Non-fuzzy variable coefficients are real-valued functions, whereas fuzzy variable coefficients are represented as fuzzy-valued functions. Among the different types of fuzzy variable coefficients, triangular and trapezoidal fuzzy function coefficients are particularly notable.

Gasilov et al. (2012) introduced the concept of triangular fuzzy functions (TFFs) in the study of fuzzy functions. These researchers applied a fuzzy exact-analytical method to solve second-order linear FDEs where the coefficients are defined as TFFs. Subsequent studies on TFFs by various scholars, including Mondal and Roy (2013), Eljaoui et al. (2015), Patel and Desai (2017), Citil (2020), Alikhani and Mostafazadeh (2021), and Jamal et al. (2022), have primarily focused on linear FDEs with TFF coefficients, using fuzzy exact-analytical methods.

However, these methods have not been applied to non-linear FDEs. The solutions obtained for linear FDEs are known as fuzzy exact-analytical solutions. It is important to note that exact analytical solutions can be elusive and complex. Therefore, there is a need to explore approximate solutions when exact solutions are not feasible.

In our current work, we will employ the fuzzy homotopy analysis method (FHAM) to derive fuzzy approximate analytical solutions for FDEs characterized by TFF coefficients. Our study will address both linear and non-linear FDEs.

2. Fundamental concepts in fuzzy set theory

The basic definitions in the fuzzy theory, which are: Fuzzy set, $\alpha -$level set, fuzzy number, fuzzy function, fuzzy derivative, etc., are found in detail in Wang and Guo (2011), Citil (2019), Sabr et al. (2021), and Suhhiem and Khwayyit (2022). In this section, we will touch on definitions that are directly related to our work.

Definition 2.1: Triangular fuzzy number: A triangular fuzzy number is a fuzzy number represented with three points as follows (Gasilov et al., 2012):

$\tilde{A} = (a, b, c)$, with $a \le b \le c$

This representation is interpreted as a membership function.

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$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & ;\ if \quad x < a \\ \frac{x-a}{b-a} & ;\ if \quad a \le x \le b \\ \frac{c-x}{c-b} & ;\ if \quad b \le x \le c \\ 0 & ;\ if \quad x > c \end{cases} \tag{1}$$

Remark 2.2: The parametric form of the triangular fuzzy number $\tilde{A}=(a\,,b\,,c)$ is (Gasilov et al., 2012):

$$[A]_\alpha = [\underline{A}\ ,\overline{A}\,] = [\,(b-a)\,\alpha + a\ ,(b-c)\,\alpha + c\ ] \tag{2}$$

Definition 2.3: Triangular fuzzy function: Let $F_a\,, F_b$ and $F_c : I \rightarrow R$ for some interval $I \subseteq R$ be a continuous real-valued function with (Gasilov et al., 2012):

$F_a(x) \le F_b(x) \le F_c(x)\,,\ \forall\, x \in I$.

We call the fuzzy set $\tilde{F}$, determined by the membership function:

$$\mu_{\tilde{F}}(y) = \begin{cases} \alpha\,, if \quad y = (F_b - F_a)\alpha + F_a \quad and \quad 0 < \alpha \le 1 \\ \alpha\,, if \quad y = (F_b - F_c)\alpha + F_c \quad and \quad 0 < \alpha \le 1 \\ 0\,, \quad otherwise \end{cases} \tag{3}$$

as TFF, and it is denoted by:

$\tilde{F} = (F_a\,,\ F_b\,,\ F_c\,)$

According to this definition, a TFF is a fuzzy set of real-valued functions, and it is a special kind of fuzzy function.

Remark 2.4: The parametric form of the TFF $\tilde{F} = (F_a\,,\ F_b\,,\ F_c\,)$ is (Gasilov et al., 2012):

$$\tilde{F} = [\,F\,]_\alpha = [\underline{F}\ ,\overline{F}\,] = [(F_b - F_a)\alpha + F_a\ ,(F_b - F_c)\alpha + F_c] \tag{4}$$

3. Fuzzy differential equations with triangular fuzzy function coefficients

The general form of the nth-order FDE with TFF coefficients is:

$$x^{(n)}(t\,;\alpha) + \tilde{a}_{n-1}\,x^{(n-1)}(t\,;\alpha) + \tilde{a}_{n-2}\,x^{(n-2)}(t\,;\alpha) + \cdots + \tilde{a}_1\,x'(t\,;\alpha) + \tilde{a}_0 x(t\,;\alpha) = \tilde{k} \tag{5}$$

with:

$$x(0\,;\alpha) = b_0(\alpha),\ x'(0\,;\alpha) = b_1(\alpha)\ ,\ x''(0\,;\alpha) = b_2(\alpha),\ldots,x^{(n-1)}(0\,;\alpha) = b_{n-1}(\alpha) \tag{6}$$

where, $\tilde{a}_{n-1}\,,\ \tilde{a}_{n-2}\,,\ldots\,,\ \tilde{a}_1\,,\ \tilde{a}_0$ and $\tilde{k}$ are TFF; $b_0(\alpha)\,,\ b_1(\alpha)\,,\ b_2(\alpha)\,,\ldots,b_{n-1}(\alpha)$ are fuzzy numbers; $x(t\,;\alpha)$ is the fuzzy solution to be determined.

Since $\tilde{a}_{n-1}\,,\ \tilde{a}_{n-2}\,,\ldots\,,\ \tilde{a}_1\,,\ \tilde{a}_0$ and $\tilde{k}$ are TFF, then we must have:

$$\tilde{a}_{n-1} = (\,g_1(t)\,,\ g_2(t)\,,\ g_3(t)\,) \tag{7}$$
$$\tilde{a}_{n-2} = (\,g_4(t)\,,\ g_5(t)\,,\ g_6(t)\,) \tag{8}$$
$$\vdots$$
$$\tilde{a}_1 = (\,g_7(t)\,,\ g_8(t)\,,\ g_9(t)\,) \tag{9}$$
$$\tilde{a}_0 = (\,g_{10}(t)\,,\ g_{11}(t)\,,\ g_{12}(t)\,) \tag{10}$$
$$\tilde{k} = (\,g_{13}(t)\,,\ g_{14}(t)\,,\ g_{15}(t)\,) \tag{11}$$

where, $g_i(t) : I \rightarrow R$, $i=1, 2, 3, \ldots$ is a continuous real-valued function on some interval $I \subseteq R$.

From the remarks 2.2 and 2.4, we can find the parametric form of the Eqs. 7-11 as follows:

$$\tilde{a}_{n-1} = [\,a_{n-1}\,]_\alpha = [\,\underline{a_{n-1}}\ ,\overline{a_{n-1}}\,] = [\,(\,g_2(t) - g_1(t))\alpha + g_1(t)\ ,(\,g_2(t) - g_3(t))\alpha + g_3(t)] \tag{12}$$
$$\tilde{a}_{n-2} = [\,a_{n-2}\,]_\alpha = [\,\underline{a_{n-2}}\ ,\overline{a_{n-2}}\,] = [\,(\,g_5(t) - g_4(t))\alpha + g_4(t)\ ,(\,g_5(t) - g_6(t))\alpha + g_6(t)] \tag{13}$$
$$\vdots$$
$$\tilde{a}_1 = [\,a_1\,]_\alpha = [\,\underline{a_1}\ ,\overline{a_1}\,] = [\,(\,g_8(t) - g_7(t))\alpha + g_7(t)\ ,(\,g_8(t) - g_9(t))\alpha + g_9(t)] \tag{14}$$
$$\tilde{a}_0 = [\,a_0\,]_\alpha = [\,\underline{a_0}\ ,\overline{a_0}\,] = [\,(\,g_{11}(t) - g_{10}(t))\alpha + g_{10}(t), (\,g_{11}(t) - g_{12}(t))\alpha + g_{12}(t)] \tag{15}$$
$$\tilde{k} = [\,k\,]_\alpha = [\,\underline{k}\ ,\overline{k}\,] = [\,(\,g_{14}(t) - g_{13}(t))\alpha + g_{13}(t)\ ,(\,g_{14}(t) - g_{15}(t))\alpha + g_{15}(t)] \tag{16}$$

To solve the problem in Eq. 5, we convert it into a system of nth-order non-fuzzy differential equations as follows:

$$[\,x^{(n)}(t) + a_{n-1}\,x^{(n-1)}(t\,) + a_{n-2}\,x^{(n-2)}(t\,) + \cdots + a_1\,x'(t) + a_0\,x(t)]_\alpha = [\,k\,]_\alpha \tag{17}$$

with:

$$[\,x^{(j)}(0)]_\alpha = [\,b_j]_\alpha,\ 0 \le j \le n-1 \tag{18}$$

Then we get:

$$[\,x^{(n)}(t)\,]_\alpha + [a_{n-1}\,x^{(n-1)}(t\,)]_\alpha + [a_{n-2}\,x^{(n-2)}(t\,)]_\alpha + \cdots + [a_1 x'(t)]_\alpha + [a_0 x(t)]_\alpha = [\,k\,]_\alpha \tag{19}$$

Which give:

$$[x^{(n)}(t)]_\alpha + [a_{n-1}]_\alpha[x^{(n-1)}(t\,)]_\alpha + [a_{n-2}]_\alpha[x^{(n-2)}(t\,)]_\alpha + \cdots + [a_1]_\alpha[x'(t)]_\alpha + [a_0]_\alpha[x(t)]_\alpha = [\,k\,]_\alpha \tag{20}$$

In parametric form, the problem 20 will become:

$$[[\,x^{(n)}(t)]_\alpha^L, [\,x^{(n)}(t)]_\alpha^U] + [[a_{n-1}(t)]_\alpha^L, [a_{n-1}(t)]_\alpha^U][[x^{(n-1)}(t\,)]_\alpha^L, [x^{(n-1)}(t\,)]_\alpha^U] + \cdots + [[\,a_1(t)]_\alpha^L, [\,a_1(t)]_\alpha^U\,]\,[\,[x'(t\,)]_\alpha^L, [x'(t\,)]_\alpha^U\,] + [[\,a_0(t)]_\alpha^L, [\,a_0(t)]_\alpha^U\,]\,[[x(t)]_\alpha^L, [x(t)]_\alpha^U\,] = [\,[k(t)]_\alpha^L, [k(t)]_\alpha^U\,] \tag{21}$$

with:

$$[[x^{(j)}(0)]_\alpha^L\ ,[x^{(j)}(0)]_\alpha^U\,] = [[\,b_j]_\alpha^L\ ,[\,b_j]_\alpha^U];\ 0 \le j \le n-1 \tag{22}$$

where,

$$[\,a_{n-1}(t)]_\alpha^L = (\,g_2(t) - g_1(t))\alpha + g_1(t) \tag{23}$$
$$[\,a_{n-1}(t)]_\alpha^U = (\,g_2(t) - g_3(t))\alpha + g_3(t) \tag{24}$$
$$[\,a_{n-2}(t)]_\alpha^L = (\,g_5(t) - g_4(t))\alpha + g_4(t) \tag{25}$$
$$[\,a_{n-2}(t)]_\alpha^U = (\,g_5(t) - g_6(t))\alpha + g_6(t) \tag{26}$$
$$\vdots$$
$$[\,a_1(t)]_\alpha^L = (\,g_8(t) - g_7(t))\alpha + g_7(t) \tag{27}$$
$$[\,a_1(t)]_\alpha^U = (\,g_8(t) - g_9(t))\alpha + g_9(t) \tag{28}$$
$$[\,a_0(t)]_\alpha^L = (\,g_{11}(t) - g_{10}(t))\alpha + g_{10}(t) \tag{29}$$

$$[a_0(t)]_\alpha^U = (g_{11}(t) - g_{12}(t))\alpha + g_{12}(t) \quad (30)$$
$$[k(t)]_\alpha^L = (g_{14}(t) - g_{13}(t))\alpha + g_{13}(t) \quad (31)$$
$$[k(t)]_\alpha^U = (g_{14}(t) - g_{15}(t))\alpha + g_{15}(t) \quad (32)$$

Then we have a system of nth-order non-fuzzy differential equations:

$$[x^{(n)}(t)]_\alpha^L + [a_{n-1}(t)]_\alpha^L [x^{(n-1)}(t)]_\alpha^L + [a_{n-2}(t)]_\alpha^L [x^{(n-2)}(t)]_\alpha^L + \cdots + [a_1(t)]_\alpha^L [x'(t)]_\alpha^L + [a_0(t)]_\alpha^L [x(t)]_\alpha^L = [k(t)]_\alpha^L \quad (33)$$
$$[x^{(n)}(t)]_\alpha^U + [a_{n-1}(t)]_\alpha^U [x^{(n-1)}(t)]_\alpha^U + [a_{n-2}(t)]_\alpha^U [x^{(n-2)}(t)]_\alpha^U + \cdots + [a_1(t)]_\alpha^U [x'(t)]_\alpha^U + [a_0(t)]_\alpha^U [x(t)]_\alpha^U = [k(t)]_\alpha^U \quad (34)$$

with:

$$[x^{(j)}(0)]_\alpha^L = [b_j]_\alpha^L \quad , 0 \le j \le n-1 \quad (35)$$
$$[x^{(j)}(0)]_\alpha^U = [b_j]_\alpha^U \quad , 0 \le j \le n-1 \quad (36)$$

By solving the Eqs. 33 and 34, we can obtain the fuzzy solution of problem 5:

$$x(t;\alpha) = [x(t)]_\alpha = [\,[x(t)]_\alpha^L ,\ [x(t)]_\alpha^U\,] \quad (37)$$

where,

$[x(t)]_\alpha^L$ is the lower bound of the fuzzy solution;
$[x(t)]_\alpha^U$ is the upper bound of the fuzzy solution.

In the same way, the above mathematical description can be repeated if the FDE is a boundary value problem or non-linear.

4. Fuzzy homotopy analysis method

To describe FHAM, we consider the FDE (Sabr et al., 2021):

$$[F(x(t)]_\alpha = 0 \quad (38)$$

where, F is the non-linear operator, t denotes the independent non-fuzzy variable, $x(t)$ is the fuzzy solution that must be computed.

The fuzzy deformation equation of order 0 is:

$$[(1-v)L(u(t;v) - x_0(t))]_\alpha = [vcF(u(t;v))]_\alpha \quad (39)$$

where, $v \in [0,1]$, $c \in [-1,0)$ is the convergence control parameter, L is the linear operator, $x_0(t)$ is the fuzzy initial guess of $x(t)$ and $u(t;v)$ is a fuzzy function.

When $v = 0$ and $v = 1$, then we have:

$$[u(t;0)]_\alpha = [x_0(t)]_\alpha \quad (40)$$
$$[u(t;1)]_\alpha = [x(t)]_\alpha \quad (41)$$

Therefore, when v is increasing from 0 to 1, the fuzzy solution $[u(t;v)]_\alpha$ varies from the fuzzy initial guess $[x_0(t)]_\alpha$ to the fuzzy solution$[x(t)]_\alpha$

By expanding $[u(t;v)]_\alpha$ in the Taylor series, with respect to v, we get:

$$[u(t;v)]_\alpha = [x_0(t)]_\alpha + \sum_{m=1}^{\infty}[x_m(t)v^m]_\alpha \quad (42)$$

where,

$$[x_m(t)]_\alpha = \frac{1}{m!}\frac{\partial^m [u(t;v)]_\alpha}{\partial v^m}\bigg|_{v=0} \quad (43)$$

If the parameter c is so properly chosen, then the fuzzy series in Eq. 42 converges at $v = 1$ and we get:

$$[u(t;1)]_\alpha = [x(t)]_\alpha = [x_0(t)]_\alpha + \sum_{m=1}^{\infty}[x_m(t)]_\alpha \quad (44)$$

which must be one of the fuzzy solutions of the problem in Eq. 38.

Now, if $c = -1$, the Eq. 39 will be:

$$[(1-v)L(u(t;v) - x_0(t))]_\alpha + [vF(u(t;v))]_\alpha = 0 \quad (45)$$

which is used mostly in FHAM.

Now, we define the fuzzy vectors:

$$[\vec{x}_i]_\alpha = \{[x_0(t)]_\alpha ,\ [x_1(t)]_\alpha , [x_2(t)]_\alpha , \ldots , [x_i(t)]_\alpha\} \quad (46)$$

Thus, by applying some mathematical operations on Eq. 39, we have the mth-order fuzzy deformation equation:

$$L([x_m(t)]_\alpha - \mu_m[x_{m-1}(t)]_\alpha) = c([R_m(\vec{x}_{m-1})]_\alpha) \quad (47)$$

where,

$$[R_m(\vec{x}_{m-1})]_\alpha = \frac{1}{(m-1)!}\frac{\partial^{m-1}[F(u(t;v))]_\alpha}{\partial v^{m-1}}\bigg|_{v=0} \quad (48)$$
$$\mu_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \quad (49)$$

From the above-mentioned mathematical description, we can conclude that the fuzzy series (approximate) solution of the problem 38 is:

$$[x(t)]_\alpha = [\,[x(t)]_\alpha^L , [x(t)]_\alpha^U\,] \quad (50)$$

where,

$$[x(t)]_\alpha^L = [x_0(t)]_\alpha^L + [x_1(t)]_\alpha^L + [x_2(t)]_\alpha^L + [x_3(t)]_\alpha^L + \cdots \quad (51)$$
$$[x(t)]_\alpha^U = [x_0(t)]_\alpha^U + [x_1(t)]_\alpha^U + [x_2(t)]_\alpha^U + [x_3(t)]_\alpha^U + \cdots \quad (52)$$

where, the initial approximations $[x_0(t)]_\alpha^L$ and $[x_0(t)]_\alpha^U$ can be found by using the Taylor series expansion of the initial conditions.

While the rest terms $[x_1(t)]_\alpha^L$, $[x_1(t)]_\alpha^U$, $[x_2(t)]_\alpha^L$, $[x_2(t)]_\alpha^U$, ... can be obtained by applying the Eqs. 42, 47, 48, and 49.

It is necessary to note that: $L(\cdot) = \frac{d^n}{dt^n}(\cdot)$ For the nth-order FDE and thus its inverse L^{-1} follows as the n-times definite integration operator from t_0 to t.

5. Applied examples

In this section, four fuzzy problems have been solved. We test the accuracy of the obtained solutions by computing the absolute errors:

$$[error]_\alpha^L = |\,[x_{exact}(t)]_\alpha^L - [x_{series}(t)]_\alpha^L\,|$$
$$[error]_\alpha^U = |\,[x_{exact}(t)]_\alpha^U - [x_{series}(t)]_\alpha^U\,|$$

Example 1: Consider the linear first order FDE:

$x'(t) = (0.45, 0.50, 0.60)x(t) + (t, 2t, 5t), t \in [0, 0.5]$

with:

$x(0) = (0.14, \quad 0.17, \quad 0.23)$

Solution: The parametric form of this problem can be written as:

$[x'(t)]_\alpha = [0.05\,\alpha + 0.45, -0.10\,\alpha + 0.60][x(t)]_\alpha + [\alpha t + t, -3\alpha t + 5t]$

with:

$[x(0)]_\alpha = [0.03\,\alpha + 0.14, -0.06\,\alpha + 0.23]$

Then, we get the following system:

$[x'(t)]_\alpha^L = (0.05\alpha + 0.45)[x(t)]_\alpha^L + (\alpha t + t)$
$[x'(t)]_\alpha^U = (-0.10\,\alpha + 0.60)[x(t)]_\alpha^U + (-3\alpha t + 5t)$

with:

$[x(0)]_\alpha^L = 0.03\alpha + 0.14$
$[x(0)]_\alpha^U = -0.06\alpha + 0.23$

The fuzzy linear operator is:

$[L(u(t;v))]_\alpha = [\,[L(u(t;v))]_\alpha^L, [L(u(t;v))]_\alpha^U\,]$

where,

$[L(u(t;v))]_\alpha^L = \left[\frac{\partial u(t;v)}{\partial t}\right]_\alpha^L$
$[L(u(t;v))]_\alpha^U = \left[\frac{\partial u(t;v)}{\partial t}\right]_\alpha^U$

We define the fuzzy non-linear operator as:

$[F(u(x;v))]_\alpha = [\,[F(u(x;v))]_\alpha^L, [F(u(x;v))]_\alpha^U\,]$

where,

$[F(u(x;v))]_\alpha^L = \frac{\partial [u(t;v)]_\alpha^L}{\partial t} - (0.05\alpha + 0.45)[u(t;v)]_\alpha^L - (\alpha t + t)$
$[F(u(x;v))]_\alpha^U = \frac{\partial [u(t;v)]_\alpha^U}{\partial t} - (-0.1\alpha + 0.6)[u(t;v)]_\alpha^U - (-3\alpha t + 5t)$

Now, we apply the mathematical steps in section 4 to get:

$[x_0(t)]_\alpha^L = 0.03\alpha + 0.14$
$[x_0(t)]_\alpha^U = -0.06\alpha + 0.23$
$[x_1(t)]_\alpha^L = -0.0015\alpha^2 c\,t - 0.0205\,\alpha c\,t - 0.063\,c\,t - 0.5\alpha c\,t^2 - 0.5c\,t^2$
$[x_1(t)]_\alpha^U = -0.006\alpha^2 c\,t + 0.0590\,\alpha c\,t - 0.138\,c\,t + 1.5\alpha c\,t^2 - 2.5c\,t^2$
$[x_2(t)]_\alpha^L = -0.0015\,\alpha^2 c^2 t - 0.0205\,\alpha c^2 t - 0.063\,c^2 t - 0.493813\alpha c^2 t^2 - 0.485825\,c^2 t^2 + 0.000038\,\alpha^3 c^2 t^2 + 0.00085\,\alpha^2 c^2 t^2 + 0.008333\,\alpha^2 c^2 t^3 + 0.083333\,\alpha c^2 t^3 + 0.075\,c^2 t^3 - 0.0015\,\alpha^2 c\,t - 0.0205\,\alpha c\,t - 0.063\,c\,t - 0.5\alpha\,c\,t^2 - 0.5\,c\,t^2$
$[x_2(t)]_\alpha^U = -0.006\,\alpha^2 c^2 t + 0.059\,\alpha c^2 t - 0.138\,c^2 t + 1.4754\alpha c^2 t^2 - 2.4586\,c^2 t^2 - 0.0003\,\alpha^3 c^2 t^2 + 0.00475\alpha^2 c^2 t^2 + 0.05\,\alpha^2 c^2 t^3 - 0.383333\,\alpha c^2 t^3 + 0.5\,c^2 t^3 - 0.006\,\alpha^2 c\,t + 0.059\,\alpha c\,t - 0.138\,c\,t + 1.5\,\alpha c t^2 - 2.5\,c\,t^2$
⋮

Thus, the semi-analytical solution is:

$[x(t)]_\alpha = [\,[x(t)]_\alpha^L, [x(t)]_\alpha^U\,]$

where,

$[x(t)]_\alpha^L = 0.03\,\alpha + 0.14 - 0.003\,\alpha^2 c\,t - 0.041\,\alpha c\,t - 0.126\,c\,t - \alpha c\,t^2 - c\,t^2 - 0.0015\,\alpha^2 c^2 t - 0.0205\,\alpha c^2 t - 0.063\,c^2 t - 0.493813\,\alpha c^2 t^2 - 0.485825\,c^2 t^2 + 0.000038\,\alpha^3 c^2 t^2 + 0.00085\,\alpha^2 c^2 t^2 + 0.008333\,\alpha^2 c^2 t^3 + 0.083333\,\alpha c^2 t^3 + 0.075\,c^2 t^3 + \cdots$
$[x(t)]_\alpha^U = -0.06\,\alpha + 0.23 - 0.012\,\alpha^2 c\,t + 0.118\,\alpha c\,t - 0.276\,c\,t + 3\,\alpha c\,t^2 - 5\,c\,t^2 - 0.006\,\alpha^2 c^2 t + 0.059\,\alpha c^2 t - 0.138\,c^2 t + 1.4754\,\alpha c^2 t^2 - 2.4586\,c^2 t^2 - 0.0003\,\alpha^3 c^2 t^2 + 0.00475\,\alpha^2 c^2 t^2 + 0.05\,\alpha^2 c^2 t^3 - 0.383333\,\alpha c^2 t^3 + 0.5\,c^2 t^3 + \cdots$

At $c = -1$, The solution will be:

$[x(t)]_\alpha = [\,[x(t)]_\alpha^L, [x(t)]_\alpha^U\,]$

where,

$[x(t)]_\alpha^L \simeq 0.03\alpha + 0.14 + 0.0015\,\alpha^2 t + 0.0205\,\alpha t + 0.063\,t + 0.506187\,\alpha t^2 + 0.514175\,t^2 + 0.000038\,\alpha^3 t^2 + 0.00085\,\alpha^2 t^2 + 0.008333\,\alpha^2 t^3 + 0.083333\,\alpha t^3 + 0.075\,t^3 + \cdots$
$[x(t)]_\alpha^U \simeq -0.06\alpha + 0.23 + 0.006\,\alpha^2 t - 0.059\,\alpha t + 0.138\,t - 1.5246\,\alpha t^2 + 2.5414\,t^2 - 0.0003\,\alpha^3 t^2 + 0.00475\,\alpha^2 t^2 + 0.05\,\alpha^2 t^3 - 0.383333\,\alpha t^3 + 0.5\,t^3 + \cdots$

The fuzzy exact-analytical solution at $\alpha = 0$ is:

$[x(t)]_\alpha = [\,[x(t)]_\alpha^L, [x(t)]_\alpha^U\,]$

where,

$[x(t)]_\alpha^L = \frac{-20}{9}t - \frac{400}{81} + \frac{20567}{4050}e^{0.45t}$
$[x(t)]_\alpha^U = \frac{-25}{3}t - \frac{125}{9} + \frac{12707}{900}e^{0.60t}$

The fuzzy series solution that we have obtained at $\alpha = 0$ is:

$[x(t)]_\alpha = [\,[x(t)]_\alpha^L, [x(t)]_\alpha^U\,]$

where,

$[x(t)]_\alpha^L \simeq 0.14 + 0.063\,t + 0.514175\,t^2 + 0.075\,t^3 + \cdots$
$[x(t)]_\alpha^U \simeq 0.23 + 0.138\,t + 2.54140\,t^2 + 0.5\,t^3 + \cdots$

Numerical results for this problem can be found in Table 1.

Example 2: Consider the non-linear first order FDE:

$x'(t) = (0.5, 1, 1.5)\,x^2(t) + (0.75, 1, 1.25), t \in [0, 0.1]$

with:

$x(0) = (0,\ 0,\ 0)$

Solution: The parametric form of this problem can be written as:

$[x'(t)]_\alpha = [0.5\,\alpha + 0.5\,, -0.5\,\alpha + 1.5][x^2(t)]_\alpha +$
$[0.25\alpha + 0.75\,, -0.25\alpha + 1.25]$

with:

$[x(0)]_\alpha = [0\,, 0]$

Then, we get the following system:

$[x'(t)]_\alpha^L = (0.5\alpha + 0.5)[x^2(t)]_\alpha^L + (0.25\alpha + 0.75)$
$[x'(t)]_\alpha^U = (-0.5\alpha + 1.5)[x^2(t)]_\alpha^U + (-0.25\alpha + 1.25)$

with:

$[x(0)]_\alpha^L = 0$
$[x(0)]_\alpha^U = 0$

The fuzzy linear operator is:

$[L(u(t;v))]_\alpha = [\,[L(u(t;v))]_\alpha^L\,, [L(u(t;v))]_\alpha^U\,]$

where,

$[L(u(t;v))]_\alpha^L = \left[\frac{\partial u(t;v)}{\partial t}\right]_\alpha^L$
$[L(u(t;v))]_\alpha^U = \left[\frac{\partial u(t;v)}{\partial t}\right]_\alpha^U$

We define the fuzzy non-linear operator as:

$[F(u(x;v))]_\alpha = [\,[F(u(x;v))]_\alpha^L\,, [F(u(x;v))]_\alpha^U\,]$

where,

$[F(u(x;v))]_\alpha^L = \frac{\partial [u(t;v)]_\alpha^L}{\partial t} - (0.5\alpha + 0.5)[(u(t;v))^2]_\alpha^L -$
$(0.25\alpha + 0.75)$
$[F(u(x;v))]_\alpha^U = \frac{\partial [u(t;v)]_\alpha^U}{\partial t} - (-0.5\alpha + 1.5)[(u(t;v))^2]_\alpha^U -$
$(-0.25\alpha + 1.25)$

Now, we apply the mathematical steps in section 4 to get:

$[x_0(t)]_\alpha^L = 0$
$[x_0(t)]_\alpha^U = 0$
$[x_1(t)]_\alpha^L = -0.25\,\alpha c\,t - 0.75\,c\,t$
$[x_1(t)]_\alpha^U = 0.25\,\alpha c\,t - 1.25\,c\,t$
$[x_2(t)]_\alpha^L = -0.25\,\alpha c^2 t - 0.75\,c^2 t - 0.25\,\alpha c\,t - 0.75\,c\,t$
$[x_2(t)]_\alpha^U = 0.25\,\alpha c^2 t - 1.25\,c^2 t + 0.25\,\alpha c\,t - 1.25\,c\,t$
$[x_3(t)]_\alpha^L = -0.25\,\alpha c^3 t - 0.75\,c^3 t - 0.5\,\alpha c^2 t - 1.5\,c^2 t -$
$0.010417\,\alpha^3 c^3 t^3 - 0.135417\,\alpha^2 c^3 t^3 - 0.09375\,\alpha c^3 t^3 -$
$0.09375\,c^3 t^3 - 0.25\,\alpha c\,t - 0.75\,c\,t$
$[x_3(t)]_\alpha^U = 0.25\,\alpha c^3 t - 1.25\,c^3 t + 0.5\,\alpha c^2 t - 2.5\,c^2 t +$
$0.010417\,\alpha^3 c^3 t^3 - 0.135417\,\alpha^2 c^3 t^3 + 0.572917\,\alpha c^3 t^3 -$
$0.78125\,c^3 t^3 + 0.25\,\alpha c\,t - 1.25\,c\,t$
$\vdots$

Thus, the semi-analytical solution is:

$[x(t)]_\alpha = [\,[x(t)]_\alpha^L\,, [x(t)]_\alpha^U\,]$

where,

$[x(t)]_\alpha^L = -0.75\,\alpha c\,t - 2.25\,c\,t - 0.25\,\alpha c^2 t - 0.75\,c^2 t -$
$0.25\,\alpha c^3 t - 0.75\,c^3 t - 0.5\,\alpha c^2 t - 1.5\,c^2 t -$
$0.010417\,\alpha^3 c^3 t^3 - 0.135417\,\alpha^2 c^3 t^3 - 0.093750\,\alpha c^3 t^3 -$
$0.093750\,c^3 t^3 + \cdots$
$[x(t)]_\alpha^U = 0.75\,\alpha c\,t - 3.75\,c\,t + 0.25\,\alpha c^2 t - 1.25\,c^2 t +$
$0.25\,\alpha c^3 t - 1.25\,c^3 t + 0.5\,\alpha c^2 t - 2.5\,c^2 t +$
$0.010417\,\alpha^3 c^3 t^3 - 0.135417\,\alpha^2 c^3 t^3 + 0.572917\,\alpha c^3 t^3 -$
$0.781250\;c^3 t^3 + \cdots$

At $c = -1$, The solution will be:

$[x(t)]_\alpha = [\,[x(t)]_\alpha^L\,, [x(t)]_\alpha^U\,]$

where,

$[x(t)]_\alpha^L \simeq 0.25\,\alpha t + 0.75\,t + 0.010417\,\alpha^3 t^3 +$
$0.135417\,\alpha^2 t^3 + 0.093750\,\alpha t^3 + 0.093750\,t^3 + \cdots$
$[x(t)]_\alpha^U \simeq -0.25\,\alpha t + 1.25\,t - 0.010417\,\alpha^3 t^3 +$
$0.135417\,\alpha^2 t^3 - 0.572917\,\alpha t^3 + 0.781250\,t^3 + \cdots$

The fuzzy exact-analytical solution at $\alpha = 0.5$ is:

$[x(t)]_\alpha = [\,[x(t)]_\alpha^L\,, [x(t)]_\alpha^U\,]$

where,

$[x(t)]_\alpha^L = \sqrt{\frac{7}{6}}\;tan(\sqrt{\frac{21}{32}}\;t\,)$
$[x(t)]_\alpha^U = \sqrt{\frac{9}{10}}\;tan(\sqrt{\frac{45}{32}}\;t\,)$

The fuzzy series solution that we have obtained at $\alpha = 0.5$ is:

$[x(t)]_\alpha = [\,[x(t)]_\alpha^L\,, [x(t)]_\alpha^U\,]$

where,

$[x(t)]_\alpha^L = 0.8750\,t\; + 0.175781375\,t^3\;\; + \cdots$
$[x(t)]_\alpha^U = \;1.1250\,t\; + 0.527343625\,t^3\;\; + \cdots$

Numerical results for this problem can be found in Table 2.

Example 3: Consider the linear second order FDE:

$x''(t) - (0, 2, 3\,)\,x'(t) = (4t^2, 5t^2\,, 7t^2\,), t \in [0\,, 0.25]$

with:

$x(0) = (1\,, 2, 2.5\,)\;,\; x'(0) = (3.5\,, 4\,, 4.5)$

Solution: The parametric form of this problem can be written as:

$[x''(t)]_\alpha - [2\alpha, -\alpha + 3][x'(t)]_\alpha = [\alpha t^2 + 4t^2,\; -2\alpha t^2 +$
$7t^2]$

with:

$[x(0)]_\alpha = [\alpha + 1\,, -0.5\alpha + 2.5]$
$[x'(0)]_\alpha = [0.5\alpha + 3.5\,, -0.5\alpha + 4.5]$

Then, we get the following system:

$[x''(t)]_\alpha^L - (2\,\alpha)[x'(t)]_\alpha^L = \alpha t^2 + 4t^2$
$[x''(t)]_\alpha^U - (-\,\alpha + 3)[x'(t)]_\alpha^U = -2\alpha t^2 + 7t^2$

with:

$[x(0)]_\alpha^L = \alpha + 1$, $[x'(0)]_\alpha^L = 0.5\alpha + 3.5$
$[x(0)]_\alpha^U = -0.5\alpha + 2.5$, $[x'(0)]_\alpha^U = -0.5\alpha + 4.5$

The fuzzy linear operator is:

$$[L(u(t;v))]_\alpha = [\,[L(u(t;v))]_\alpha^L , [L(u(t;v))]_\alpha^U\,]$$

where,

$$[L(u(t;v))]_\alpha^L = \left[\frac{\partial^2 u(t;v)}{\partial t^2}\right]_\alpha^L$$
$$[L(u(t;v))]_\alpha^U = \left[\frac{\partial^2 u(t;v)}{\partial t^2}\right]_\alpha^U$$

We define the fuzzy non-linear operator as:

$$[F(u(x;v))]_\alpha = [\,[F(u(x;v))]_\alpha^L , [F(u(x;v))]_\alpha^U\,]$$

where,

$$[F(u(x;v))]_\alpha^L = \frac{\partial^2 [u(t;v)]_\alpha^L}{\partial t^2} + (2\alpha)\frac{\partial [u(t;v)]_\alpha^L}{\partial t} - \alpha t^2 - 4t^2$$
$$[F(u(x;v))]_\alpha^U = \frac{\partial^2 [u(t;v)]_\alpha^U}{\partial t^2} + (-\alpha + 3)\frac{\partial [u(t;v)]_\alpha^U}{\partial t} + 2\alpha t^2 - 7t^2$$

Now, we apply the mathematical steps in section 4 to get:

$[x_0(t)]_\alpha^L = \alpha + 1 + 0.5\,\alpha t + 3.5\,t$
$[x_0(t)]_\alpha^U = -0.5\alpha + 2.5 - 0.5\,\alpha t + 4.5\,t$
$[x_1(t)]_\alpha^L = 0.5\,\alpha^2 c\,t^2 + 3.5\alpha c\,t^2 - 0.083333\alpha c t^4 - 0.333333 c t^4$
$[x_1(t)]_\alpha^U = 0.25\alpha^2 c\,t^2 - 3\alpha c t^2 + 6.75 c t^2 + 0.166667\alpha c t^4 - 0.583333 c t^4$
$[x_2(t)]_\alpha^L = 0.5\,\alpha^2 c^2 t^2 + 3.5\,\alpha c^2 t^2 - 0.083333\alpha c^2 t^4 - 0.333333 c^2 t^4 + 0.333333\alpha^3 c^2 t^3 + 2.333333\alpha^2 c^2 t^3 - 0.033333\alpha^2 c^2 t^5 - 0.133333\alpha c^2 t^5 + 0.5\,\alpha^2 c t^2 + 3.5\alpha c t^2 - 0.083333\,\alpha c t^4 - 0.333333\,c t^4$,
$[x_2(t)]_\alpha^U = 0.25\,\alpha^2 c^2 t^2 - 3\alpha\,c^2 t^2 + 6.75\,c^2 t^2 + 0.166667\alpha c^2 t^4 - 0.583333 c^2 t^4 - 0.083333\alpha^3 c^2\,t^3 + 1.25\alpha^2 c^2 t^3 - 5.25\alpha\,c^2 t^3 - 0.033333\,\alpha^2 c^2 t^5 + 0.216667\,\alpha c^2 t^5 + 6.75\,c^2 t^3 - 0.35\,c^2 t^5 + 0.25\,\alpha^2 c\,t^2 - 3\alpha c\,t^2 + 6.75\,c\,t^2 + 0.166667\alpha c\,t^4 - 0.583333\,c\,t^4$
⋮

Thus, the semi-analytical solution is:

$$[x(t)]_\alpha = [\,[x(t)]_\alpha^L , [x(t)]_\alpha^U\,]$$

where,

$[x(t)]_\alpha^L = \alpha + 1 + 0.5\,\alpha\,t + 3.5\,t + \alpha^2 c\,t^2 + 7\alpha c\,t^2 - 0.166666\,\alpha c\,t^4 - 0.666666\,c\,t^4 + 0.5\,\alpha^2 c^2 t^2 + 3.5\,\alpha\,c^2 t^2 - 0.083333\,\alpha\,c^2 t^4 - 0.333333\,c^2 t^4 + 0.333333\,\alpha^3 c^2 t^3 + 2.333333\,\alpha^2 c^2 t^3 - 0.033333\,\alpha^2 c^2 t^5 - 0.133333\,\alpha\,c^2 t^5 + \cdots$
$[x(t)]_\alpha^U = -0.5\alpha + 2.5 - 0.5\,\alpha t + 4.5t + 0.5\alpha^2 c t^2 - 6\alpha c t^2 + 13.5 c t^2 + 0.333334\alpha c t^4 - 1.166666 c t^4 + 0.25\,\alpha^2 c^2 t^2 - 3\alpha c^2 t^2 + 6.75 c^2 t^2 + 0.166667\alpha c^2 t^4 - 0.583333\,c^2 t^4 - 0.083333\,\alpha^3 c^2 t^3 + 1.25\,\alpha^2 c^2 t^3 - 5.25\,\alpha c^2 t^3 - 0.033333\,\alpha^2 c^2 t^5 + 0.216667\alpha c^2 t^5 + 6.75\,c^2 t^3 - 0.35\,c^2 t^5 + \cdots$

At $c = -1$, The solution will be:

$$[x(t)]_\alpha = [\,[x(t)]_\alpha^L , [x(t)]_\alpha^U\,]$$

where,

$[x(t)]_\alpha^L \simeq \alpha + 1 + 0.5\,\alpha t + 3.5\,t - 0.5\,\alpha^2 t^2 - 3.5\,\alpha t^2 + 0.083333\,\alpha t^4 + 0.333333\,t^4 + 0.333333\,\alpha^3 t^3 + 2.333333\,\alpha^2 t^3 - 0.033333\,\alpha^2 t^5 - 0.133333\,\alpha t^5 + \cdots$
$[x(t)]_\alpha^U \simeq -0.5\alpha + 2.5 - 0.5\,\alpha t + 4.5\,t - 0.25\,\alpha^2 t^2 + 3\,\alpha t^2 - 6.75\,t^2 - 0.166667\,\alpha t^4 + 0.583333\,t^4 - 0.083333\,\alpha^3 t^3 + 1.25\,\alpha^2 t^3 - 5.25\,\alpha t^3 - 0.033333\,\alpha^2 t^5 + 0.216667\,\alpha t^5 + 6.75\,t^3 - 0.35\,t^5 + \cdots$

The fuzzy exact-analytical solution at $\alpha = 0.2$ is:

$$[x(t)]_\alpha = [\,[x(t)]_\alpha^L , [x(t)]_\alpha^U\,]$$

where,

$$[x(t)]_\alpha^L = -\frac{13437}{40} + \frac{2697}{8}e^{0.4t} - \frac{7}{2}t^3 - \frac{105}{4}t^2 - \frac{525}{4}t$$
$$[x(t)]_\alpha^U = \frac{58951}{96040} + \frac{34309}{19208}e^{2.8t} - \frac{11}{14}t^3 - \frac{165}{196}t^2 - \frac{825}{1372}t$$

The fuzzy series solution that we have obtained at $\alpha = 0.2$ is:

$$[x(t)]_\alpha = [\,[x(t)]_\alpha^L , [x(t)]_\alpha^U\,]$$

where,

$[x(t)]_\alpha^L = 1.2 + 3.6t - 0.72t^2 + 0.096t^3 + 0.35t^4 - 0.028t^5 + \cdots$
$[x(t)]_\alpha^U = 2.4 + 4.4t - 6.16t^2 + 5.749333333t^3 + 0.55t^4 - 0.308t^5 + \cdots$

Numerical results for this problem can be found in Table 3.

Example 4: Consider the non-linear second order FDE:

$$x''(t) + x^2(t) = (0.5\,t,\ t,\ 5\,t)\ ,\ t \in [0,1]$$

with:

$$x(0) = (2,\ 3,\ 4)\ ,\ x'(0) = (5, 6, 7)$$

Solution: The parametric form of this problem can be written as:

$$[x''(t)]_\alpha + [x^2(t)]_\alpha = [0.5\,\alpha t + 0.5\,t\ ,\ -1.5\,\alpha t + 2.5\,t]$$

with:

$[x(0)]_\alpha = [\alpha + 2, -\alpha + 4]$
$[x'(0)]_\alpha = [\alpha + 5, -\alpha + 7]$

Then, we get the following system:

$[x''(t)]_\alpha^L + [x^2(t)]_\alpha^L = 0.5\,\alpha t + 0.5\,t$
$[x''(t)]_\alpha^U + [x^2(t)]_\alpha^U = -1.5\,\alpha t + 2.5\,t$

with:

$[x(0)]_\alpha^L = \alpha + 2$, $[x'(0)]_\alpha^L = \alpha + 5$
$[x(0)]_\alpha^U = -\alpha + 4$, $[x'(0)]_\alpha^U = -\alpha + 7$

The fuzzy linear operator is:

$$[L(u(t;v))]_\alpha = [\,[L(u(t;v))]_\alpha^L , [L(u(t;v))]_\alpha^U\,]$$

where,

$[L(u(t;v))]_\alpha^L = \left[\frac{\partial u(t;v)}{\partial t}\right]_\alpha^L$
$[L(u(t;v))]_\alpha^U = \left[\frac{\partial u(t;v)}{\partial t}\right]_\alpha^U$

We define the fuzzy non-linear operator as:

$[F(u(x;v))]_\alpha = [\,[F(u(x;v))]_\alpha^L\,, [F(u(x;v))]_\alpha^U\,]$

where:

$[F(u(x;v))]_\alpha^L = \frac{\partial^2[u(t;v)]_\alpha^L}{\partial t^2} + ([u(t;v)]_\alpha^L)^2 - 0.5\,\alpha t - 0.5t$
$[F(u(x;v))]_\alpha^U = \frac{\partial^2[u(t;v)]_\alpha^U}{\partial t^2} + ([u(t;v)]_\alpha^U)^2 + 1.5\,\alpha t - 2.5t$

Now, we apply the mathematical steps in section 4 to get:

$[x_0(t)]_\alpha^L = \alpha + 2 + \ \alpha t + 5\,t$
$[x_0(t)]_\alpha^U = -\alpha + 4 - \ \alpha t + 7\,t$
$[x_1(t)]_\alpha^L = 0.5\,\alpha^2 c\,t^2 + 2\,\alpha c\,t^2 - 0.333333\alpha^2 c\,t^3 + 2.25\alpha c\,t^3 + 2c\,t^2 + 3.25\,c\,t^3 + 0.083333\,\alpha^2 c\,t^4 + 0.833333\,\alpha c\,t^4 + 2.083333\,c\,t^4$
$[x_1(t)]_\alpha^U = 0.5\,\alpha^2 c\,t^2 - 4\,\alpha c\,t^2 + 0.333333\alpha^2 c\,t^3 - 3.416667\alpha c\,t^3 + 8c\,t^2 + 8.916667c\,t^3 + 0.083333\,\alpha^2 c\,t^4 - 1.166667\alpha c\,t^4 + 4.083333\,c\,t^4$
$[x_2(t)]_\alpha^L = 0.5\,\alpha^2 c^2 t^2 + 2\alpha c^2 t^2 + 0.333333\alpha^2 c^2 t^3 + 2.25\alpha c^2 t^3 + 2c^2 t^2 + 3.25\,c^2 t^3 + 0.583333\alpha^2 c^2 t^4 + 1.833333\alpha c^2 t^4 + 2.750000c^2 t^4 + 0.083333\alpha^3 c^2 t^4 + 0.083333\alpha^3 c^2 t^5 + 0.741667\alpha^2 c^2 t^5 + 1.975\alpha c^2 t^5 + 0.027778\alpha^3 c^2 t^6 + 0.327778\alpha^2 c^2 t^6 + 1.216667\alpha c^2 t^6 + 1.65c^2 t^5 + 1.361111c^2 t^6 + 0.003968\alpha^3 c^2 t^7 + 0.059524\alpha^2 c^2 t^7 + 0.297619\alpha c^2 t^7 + 0.496032c^2 t^7 + 0.5\,\alpha^2 ct^2 + 2\alpha ct^2 + 0.333333\alpha^2 ct^3 + 2.25\alpha ct^3 + 2ct^2 + 3.25ct^3 + 0.083333\alpha^2 ct^4 + 0.833333\alpha ct^4 + 2.083333ct^4$
$[x_2(t)]_\alpha^U = 0.5\alpha^2 c^2 t^2 - 4\alpha c^2 t^2 + 0.333333\alpha^2 c^2 t^3 - 3.416667\alpha c^2 t^3 + 8\,c^2 t^2 + 8.916667c^2 t^3 + 1.083333\alpha^2 c^2 t^4 - 5.166667\alpha c^2 t^4 + 9.416666c^2 t^4 - 0.083333\alpha^3 c^2 t^4 - 0.083333\alpha^3 c^2 t^5 + 1.225\alpha^2 c^2 t^5 - 5.858334\alpha c^2 t^5 - 0.027778\alpha^3 c^2 t^6 + 0.483333\alpha^2 c^2 t^6 - 2.772223\alpha c^2 t^6 + 9.166667c^2 t^5 + 5.25c^2 t^6 - 0.003968\alpha^3 c^2 t^7 + 0.083333\alpha^2 c^2 t^7 - 0.583333\alpha c^2 t^7 + 1.361111c^2 t^7 + 0.5\,\alpha^2 ct^2 - 4\alpha ct^2 + 0.333333\alpha^2 ct^3 - 3.416667\alpha ct^3 + 8ct^2 + 8.916667ct^3 + 0.083333\alpha^2 ct^4 - 1.166667\alpha ct^4 + 4.083333ct^4$
$\vdots$

At $c = -1$, the semi-analytical solution will be:

$[x(t)]_\alpha = [\,[x(t)]_\alpha^L\,, [x(t)]_\alpha^U\,]$

where,

$[x(t)]_\alpha^L \simeq \alpha + 2 + \alpha t + 5\,t - 0.5\,\alpha^2 t^2 - 2\alpha t^2 - 2t^2 - 0.333333\,\alpha^2 t^3 - 2.25\,\alpha t^3 - 3.25\,t^3 + 0.416667\,\alpha^2 t^4 + 0.166667\,\alpha t^4 - 1.416666\,t^4 + 0.083333\,\alpha^3 t^4 + 0.083333\,\alpha^3 t^5 + 0.741667\,\alpha^2 t^5 + 1.975\,\alpha t^5 + 0.027778\,\alpha^3 t^6 + 0.327778\,\alpha^2 t^6 + 1.216667\,\alpha t^6 + 1.65\,t^5 + 1.361111\,t^6 + 0.003968\,\alpha^3 t^7 + 0.059524\,\alpha^2 t^7 + 0.297619\,\alpha t^7 + 0.496032\,t^7 + \cdots$
$[x(t)]_\alpha^U \simeq -\alpha + 4 - \alpha t + 7t - 0.5\alpha^2 t^2 + 4\alpha t^2 - 0.333333\alpha^2 t^3 + 3.416667\alpha t^3 - 8\,t^2 - 8.916667t^3 + 0.916667\alpha^2 t^4 - 2.833333\alpha t^4 + 1.25t^4 - 0.083333\alpha^3 t^4 - 0.083333\alpha^3 t^5 + 1.225\alpha^2 t^5 - 5.858334\alpha t^5 - 0.027778\alpha^3 t^6 + 0.483333\alpha^2 t^6 - 2.772223\alpha t^6 + 9.166667t^5 + 5.25t^6 - 0.003968\alpha^3 t^7 + 0.083333\alpha^2 t^7 - 0.583333\alpha t^7 + 1.361111t^7 + \cdots$

This problem has no exact analytical solution. Hence, it is clear to us the importance of FHAM, as this method provides an accurate approximate analytical solution that can be used as an effective alternative to the exact-analytical solution.

Table 1: Numerical results for example 1

t	$[x_{series}(t)]_r^L$	$[error]_r^L$	$[x_{series}(t)]_r^U$	$[error]_r^U$
0	0.14	0	0.23	0
0.00215	0.140137827519316	2.13 e-11	0.230308452590688	8.39 e-11
0.00430	0.140280413058775	1.72 e-10	0.230640430239500	6.84 e-10
0.00645	0.140427761090647	5.86 e-10	0.230995962761563	2.35 e-9
0.00860	0.140579876087200	1.40 e-9	0.231375079972000	5.68 e-9
0.01075	0.140736762520703	2.76 e-9	0.231777811685938	1.13 e-8
0.01290	0.140898424863425	4.80 e-9	0.232204187718500	1.99 e-8
0.01505	0.141064867587634	7.69 e-9	0.232654237884813	3.21 e-8
0.01720	0.141236095165600	1.16 e-8	0.233127992000000	4.88 e-8
0.01935	0.141412112069591	1.66 e-8	0.233625479879187	7.07 e-8
0.02150	0.141592922771875	2.30 e-8	0.234146731337500	9.86 e-8

Table 2: Numerical results for example 2

t	$[x_{series}(t)]_r^L$	$[error]_r^L$	$[x_{series}(t)]_r^U$	$[error]_r^U$
0	0	0	0	0
0.00125	0.001093750343323	3.05 e-11	0.001406251029968	1.15 e-15
0.00250	0.002187502746584	2.44 e-10	0.002812508239744	3.09 e-14
0.00375	0.003281259269721	8.24 e-10	0.004218777809136	2.27 e-13
0.00500	0.004375021972672	1.95 e-9	0.005625065917953	9.43 e-13
0.00625	0.005468792915375	3.82 e-9	0.007031378746002	2.86 e-12
0.00750	0.006562574157768	6.95 e-9	0.008437722473092	7.09 e-12
0.00875	0.007656367759788	1.05 e-8	0.009844103279030	1.53 e-11
0.01000	0.008750175781375	1.56 e-8	0.011250527343625	2.98 e-11
0.01125	0.009844000282466	2.23 e-8	0.012657000846685	5.36 e-11
0.01250	0.010937843322998	3.05 e-8	0.014063529968018	9.08 e-11

Table 3: Numerical results for example 3

t	$[x_{series}(t)]_r^L$	$[error]_r^L$	$[x_{series}(t)]_r^U$	$[error]_r^U$
0	1.2	0	2.4	0
0.000145	1.200521984862293	3.03 e-8	2.400637870503528	2.59 e-7
0.000290	1.201043939450344	1.21 e-7	2.401275482084224	1.04 e-6
0.000435	1.201565863765915	2.72 e-7	2.401912834847264	2.33 e-6
0.000580	1.202087757810770	4.84 e-7	2.402549928897826	4.14 e-6
0.000725	1.202609621586680	7.57 e-7	2.403186764341097	6.48 e-6
0.000870	1.203131455095417	1.09 e-6	2.403823341282268	9.33 e-6
0.001015	1.203653258338757	1.48 e-6	2.404459659826537	1.27 e-5
0.001160	1.204175031318480	1.94 e-6	2.405095720079107	1.66 e-5
0.001305	1.204696774036370	2.45 e-6	2.405731522145186	2.10 e-5
0.001450	1.205218486494215	3.03 e-6	2.406367066129991	2.59 e-5

6. Conclusion

In this work, we used FHAM to obtain the fuzzy approximate analytical solutions of FDE in which the coefficients are TFF. The approximate solutions that we obtained during this work are accurate solutions and very close to the exact solutions, based on the comparison that we made between our results and the exact solutions. This comparison was based on finding the absolute errors. Hence, the importance of the method becomes clear to us, as this method provides an accurate approximate solution that can be used as an effective alternative to the exact solution if it does not exist.

The accuracy of results depends greatly on the value of the constant c, other factors also affect, including The number of terms of the solution series, the value of α, the nature of the problem, whether linear or non-linear and the value of t. For the next works, other types of fuzzy functions can be used as coefficients of the FDE, such as trapezoidal fuzzy function and Gaussian fuzzy function.

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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