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Weighted Degree Reduction of Bézier curves with G^2 –continuity

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ABSTRACT

This paper considers weighted G^2 -degree reduction of Bézier curves. Given Bézier curves, degree reduction is an approximative process used to write it as Bézier curves of lower degree. A weight function $w[t] = 2t(1-t)$, $t \in [0, 1]$ is used in degree reducing the Bézier curves with G^2 –continuity at the end points of the curve using the L_2-norm. The boundary conditions reduce the error near the boundaries and it is anticipated that the weight function improves approximation in the middle of the curve. This is fulfilled by the numerical results and comparisons which show that the proposed method produces smaller error and outperforms existing methods.

1. Introduction

The Bézier curves possess very interesting algebraic and geometric properties, and thus, they play a fundamental role in designing and generating curves in a computer-oriented approach. These include algorithmic approach to draw curves, simplicity in evaluation and programming. They become the fundamental basis in any CAD software; see (Höllig and Hörner, 2013; Prautzsch et al., 2013). Data have to be compared, compressed, exchanged, and transferred between different CAD systems. Since different CAD systems use different degrees to represent Bézier curves, thus degree reduction has to be carried out. This issue has been investigated by many researchers. In degree reduction, in addition to the satisfaction of some conditions at the boundaries, the Bézier curve of degree n is to be approximated by a Bézier curve of degree m, $m < n$. The methods we have require finding the solution of non-linear system of equations. This suggests using numerical methods. (Lutterkort et al., 1999) proved that degree reduction of Bézier curves in the L_2 norm equals best Euclidean approximation of Bézier points, see also (Peters and Reif, 2000). These results are generalized to the constrained case by (Ahn et al., 2004), and the discrete cases have been studied in (Ait-Haddou, 2015). Rababah et al. (2007) used the idea of basis transformation between Jacobi and Bernstein to ascertain multi-degree reduction of Bézier curves. L-2 degree reduction of triangular Bézier surfaces with common tangent planes at vertices is considered in (Rababah, 2005). To find G^2-continuity conditions, we are supposed to solve nonlinear equations. The conjugate gradient method has to be utilized to solve the minimization problem. Therein, challenging difficulties are encountered; search directions lose conjugate requiring the search direction to be reset to the steepest descent direction if progress alters or stops.

The existing methods to find degree reduction have many issues including: accumulate round-off errors, stability issues, complexity, accuracy, losing conjugacy, requiring the search direction to be set to the steepest descent direction frequently, experiencing ill-conditioned systems, leading to a singularity, and the most challenging difficulty is in applying the methods (difficulty and indirect). (Rababah and Mann, 2013) presented a method to find the G^2-degree reduction and linear G^1-, G^2-, and G^3-multiple degree reduction methods for Bézier Curves. These results are expressive to researchers as well as to industrial practitioners. Their examples show that the C^2 method fails to reproduce the inner loop of the heart, while their C_1/G_2 method reproduces the loop and provides a better approximation elsewhere along the curve. The G^2-degree reduction is also studied by (Lu and Wang, 2006) and the weighted G^1-multi-degree in (Rababah and Ibrahim, 2016). Wozny and Lewanowicz (2009) studied multi-degree reduction of Bézier curves with constraints using dual Bernstein basis.

In all existing degree reducing methods, the conditions and free parameters were applied at the end points. So, there is a need to better approximate those parts close to the Centre of the curve. In this paper, we introduce a weight to take care of the Centre of the curve, it is appropriate to consider degree reduction with the weight function $w[t] = 2t$

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(1-t), $t \in [0, 1]$. The examples show that the proposed methods provide better approximation at the centre of the curves with minimum error and also reproduced these loops correctly better than existing methods.

2. Preliminaries

A Bézier curve $P_n(t)$ of degree n is defined algebraically as follows:

$$P_n(t) = \sum_{i=0}^{n} p_i B_i^n(t), \qquad 0 \le t \le 1, \qquad (1)$$

where

$$B_i^n(t) = \binom{n}{i} (1-\mathrm{t})^{\mathrm{n-i}}\, \mathrm{t^i}, \qquad i = 0,1,\dots,\mathrm{n},$$

are the Bernstein polynomials of degree n, and $P_0, P_1, \dots P_n$ are called the Bézier control points or the Bézier points, for more see (Höllig and Hörner, 2013; Prautzsch et al., 2013).

The operator Δ is defined as follows:

$$\Delta^0 p_i = p_i, \quad \Delta^k p_i = \Delta^{k-1} p_{i+1} - \Delta^{k-1} p_i, \quad k \ge 1, \qquad i = \mathrm{o}, 1, \dots, \mathrm{n-k}.$$

The first derivative of the Bézier curve is given by:

$$\frac{d}{dt} P_n(t) = n \sum_{i=0}^{n-1} \Delta p_i B_i^{n-1}(t).$$

Using the above definition of Δ, the k-th derivatives of the Bézier curve are obtained by repeating the previous process k times to get:

$$\frac{d^k}{dt^k} P_n(t) = \frac{n!}{(n-k)!} \sum_{i=0}^{n-k} \Delta^k p_i B_i^{n-k}(t).$$

A formula for the multiplication of the weight function $w(t) = 2t(1-t)$ with two Bernstein polynomials is given by:

$$B_i^m(t) B_j^n(t) 2t(1-t) = \frac{2\binom{m}{i}\binom{n}{j}}{\binom{m+n+2}{i+j+1}} B_{i+j+1}^{m+n+2}(t).$$

The integral of this defines the Gram matrix $G_{m,n}$ as $(m+1) \times (n+1)$-matrix with weight function as follows:

$$g_{ij} = \int_0^1 B_i^m(t) B_j^n(t) 2t(1-t)dt = \frac{2\binom{m}{i}\binom{n}{j}}{(m+n+3)\binom{m+n+2}{i+j+1}}, \quad i = 0,\dots,m, \ j = 0,\dots,n. \qquad (2)$$

It is clear that the matrix $G_{m,m}$ with weight function is real and symmetric. We use mathematical induction to show it is positive definite: Since the entire upper left sub matrices have positive determinants, thus, the matrix $G_{m,m}$ is a symmetric positive definite matrix, see the case in (Rababah and Mann, 2013).

3. Geometric continuity

The Bézier curves P_n and R_m are G^k-continuous at t = 0,1, see (Rababah and Mann, 2013), if there exists a strictly increasing parameterization $s(t)\colon [0,1] \to [0,1]$ with $s(0) = 0$, $s(1) = 1$, and

$$R_m^{(j)}(i) = P_n^{(j)}(s(i)), \quad i = 0,1, \quad j = 0,1,\dots,k. \qquad (3)$$

4. Degree reduction of Bézier curves

We want to find a Bézier curve $R_m(t)$ of degree m with control points $\{r_i\}_{i=0}^m$ that approximates $P_n(t)$ and satisfy the following two conditions:

(1) P_n and R_m are G^2-continuous at the end points.

(2) The weighted L_2-error between P_n and R_m is minimum.

We can write the two Bézier curves $P_n(t)$ and $R_m(t)$ in matrix form as

$$P_n(t) = \sum_{i=0}^{n} p_i B_i^n(t) =: B_n P, \quad and \ R_m(t) = \sum_{i=0}^{n} r_i B_i^m(t) =: B_m R, \qquad 0 \le t \le 1, \qquad (4)$$

In the following sections we investigate weighted degree reduction of Bézier curve with G^2-continuity at the boundaries.

5. Weighted G^2-Degree Reduction

$P_n(t)$ and $R_m(t)$ are G^2-continuous at t = 0, 1 if the two curves P_n and R_m satisfy the following conditions:

$$R_m(i) = P_n(s(i)), \qquad i = 0,1. \qquad (5)$$

$$R'_m(i) = s'(i) P'_n(s(i)), \quad s'(i) > 0, \qquad i = 0,1. \qquad (6)$$

$$R''_m(i) = (s'(i))^2 P''_n(s(i)) + s''(i) P'_n(s(i)), \qquad s'(i) > 0, \qquad i = 0,1. \qquad (7)$$

These conditions are simplified by substituting $s'(i) = \delta_i^2$, $s''(i) = \eta_i$, $i = 0,1$, $s'(i) > 0$, to get non-linear equations in δ_i; for example the last equation becomes:

$$R''_m(0) = \delta_0^4 P''_n(0) + \eta_0 P'_n(0), \qquad R''_m(1) = \delta_1^4 P''_n(1) + \eta_1 P'_n(1), \qquad (8)$$

To avoid the non-linearity, the authors in (Rababah and Mann 2013) required C^1-continuity by setting $\delta_i = 1$, $i = 0,1$ and G^2-continuity. They called this method *C_1/G_2-multi-degree reduction*. We analogously use this substitution for the case of weighted degree reduction. The following equations are obtained by substituting $\delta_0 = \delta_1 = 1$ into equations in (5) to (8) for the control points at either end point of the curve to get:

$$r_0 = p_0, \qquad r_m = p_n, \qquad (9)$$

$$r_1 = p_0 + \frac{n}{m}\Delta p_0, \qquad r_{m-1} = p_n - \frac{n}{m}\Delta p_{n-1}, \qquad (10)$$

$$r_2 = 2r_1 - r_0 + \frac{n(n-1)}{m(m-1)}\Delta^2 p_0 + \frac{n}{m(m-1)}\Delta p_0 \eta_0, \qquad (11)$$

$$r_{m-2} = 2r_{m-1} - r_m + \frac{n(n-1)}{m(m-1)}\Delta^2 p_{n-2} + \frac{n}{m(m-1)}\Delta p_{n-1}\eta_1. \qquad (12)$$

The points r_0, r_1, r_2, r_{m-2}, r_{m-1}, and r_m, are determined by G^2-continuity conditions at the boundary; accordingly, the elements of R_m can be decomposed into two parts stated as follows. The boundaries part $R_m^c = [r_0, r_1, r_2, r_{m-2}, r_{m-1}, r_m]^t$ and the interior part with interior points $R_m^f =$

$R_m \backslash R_m^c = [r_3, \dots, r_{m-3}]^t$. Similarly, B_m is decomposed in the same way into B_m^c and B_m^f.

The distance between P_n and R_m is measured using weighted L2-norm; therefore, the error term becomes

$$e = \int_0^1 ||B_n P_n - B_m R_m||^2 2t(1-t)dt$$

$$= \int_0^1 ||B_n P_n - B_m^c R_m^c - B_m^f R_m^f||^2 2t(1-t)dt. \quad (13)$$

Differentiating the error with respect to the unknown control points R_m^f we get:

$$\frac{\partial\varepsilon}{\partial R_m^f} = 2\int_0^1 \left\|B_n P_n - B_m^c R_m^c - B_m^f R_m^f\right\| B_m^f 2t(1-t)dt.$$

Evaluating the integral and equating to zero gives:

$$\frac{\partial\varepsilon}{\partial R_m^f} = G_{m,n}^p P_n - G_{m,m}^c R_m^c - G_{m,m}^f R_m^f = 0, \quad (14)$$

where

$G_{m,n}^p \coloneqq G_{m,n}(3, \dots, m-3; 0, 1, \dots, n)$, $G_{m,m}^c \coloneqq G_{m,m}(3, \dots, m-3; 0, 1, 2, m-2, m-1, m)$,

$G_{m,m}^f \coloneqq G_{m,m}(3, \dots, m-3; 3, \dots, m-3)$,

and $G_{m,n}(\dots;\dots)$ is the sub-matrix of weighted $G_{m,n}$ formed by the indicated rows and columns. Note that although we use the same notations for the matrix and submatrices of G as in (Rababah and Mann, 2013), but they do have different values and contents.

Differentiating (13) with respect to η_i, $i = 0, 1$, and equating to zero gives:

$$\frac{\partial\varepsilon}{\partial\eta_0} = (G_{m,n}^2 P_n - G_{m,m}^{2;c} R_m^c - G_{m,m}^{2;f} R_m^f).\Delta p_0 = 0, \quad (15)$$

$$\frac{\partial\varepsilon}{\partial\eta_1} = (G_{m,n}^{m-2} P_n - G_{m,m}^{m-2;c} R_m^c - G_{m,m}^{m-2;f} R_m^f).\Delta p_{n-1} = 0, \quad (16)$$

Where for $q = 2, m-2$:

$G_{m,n}^q \coloneqq G_{m,n}(q; 0, 1, \dots, n)$, $G_{m,m}^{q;c} \coloneqq G_{m,m}(q; 0, 1, 2, m-2, m-1, m)$,

$G_{m,m}^{q;f} \coloneqq G_{m,m}(q; 3, \dots, m-3)$. (17)

Note that (14) are point valued equations while (15) and (16) are scalar valued equations. Expanding (14) into its x, y, z, ... coordinates and joining them together with (15) and (16) yields a system of $d(m$ - 5) + 2 equations in $d(m$ -5) + 2 unknowns.

For the planar curves, the control points of the Bézier curve are expanded into their x and y components. Therefore, the variables of our system of equations are $r_k^x, r_k^y, k = 3, \dots, m-3$, η_0 and η_1. To express the system in a clear form, we have to decompose each of r_2 and r_{m-2} into a constant part and a part involving η_0 and η_1, respectively. Let v_2 and v_{m-2} be the constant parts of r_2 and r_{m-2} respectively. Hence

$$v_2 = 2r_1 - r_0 + \frac{n(n-1)}{m(m-1)}\Delta^2 p_0, \quad (18)$$

$$v_{m-2} = 2r_{m-1} - r_m + \frac{n(n-1)}{m(m-1)}\Delta^2 p_{n-2}. \quad (19)$$

The following vectors are defined to express the linear system in explicit form:

$P_n^C = \left[p_0^x, \dots p_n^x, p_0^y, \dots, p_n^y\right]^t$,

$R_m^F = \left[r_3^x, \dots r_{m-3}^x, r_3^y, \dots, r_{m-3}^y, \eta_0, \eta_1\right]^t$,

$R_m^C = \left[r_0^x, r_1^x, v_2^x, v_{m-2}^x, r_{m-1}^x, r_m^x, r_0^y, r_1^y, v_2^y, v_{m-2}^y, r_{m-1}^y, r_m^y\right]^t$.

Let $\oplus$ be the direct sum and define the matrices:

$G_{m,n}^{p+} = G_{m,n}^p \oplus G_{m,n}^p$, $G_{m,m}^{c+} = G_{m,m}^c \oplus G_{m,m}^c$, $G_{m,m}^{f+} = G_{m,m}^f \oplus G_{m,m}^f$.

Since the Gram matrix $G_{m,m}^f$ is real, symmetric, and positive definite, the matrix $G_{m,m}^F$ is positive definite.

Define:

$$C = \begin{bmatrix}\Delta p_0 & 0\\ 0 & \Delta p_{n-1}\end{bmatrix}\begin{bmatrix}G_{m,m}(2,2) & G_{m,m}(2,m-2)\\ G_{m,m}(m-2,2) & G_{m,m}(m-2,m-2)\end{bmatrix}\begin{bmatrix}\Delta p_0 & 0\\ 0 & \Delta p_{n-1}\end{bmatrix}$$

Further define $L_{m,n}$, $L_{m,m}^c$, $L_{m,m}^f$ as:

$$L_{m,n} = \begin{bmatrix}G_{m,n}^2\Delta p_0^x & G_{m,n}^2\Delta p_0^y\\ G_{m,n}^{m-2}\Delta p_{n-1}^x & G_{m,n}^{m-2}\Delta p_{n-1}^y\end{bmatrix}, \quad L_{m,m}^c = \begin{bmatrix}G_{m,m}^{2;c}\Delta p_0^x & G_{m,m}^{2;c}\Delta p_0^y\\ G_{m,m}^{m-2;c}\Delta p_{n-1}^x & G_{m,m}^{m-2;c}\Delta p_{n-1}^y\end{bmatrix},$$

$$L_{m,m}^f = \begin{bmatrix}G_{m,m}^{2;f}\Delta p_0^x & G_{m,m}^{2;f}\Delta p_0^y\\ G_{m,m}^{m-2;f}\Delta p_{n-1}^x & G_{m,m}^{m-2;f}\Delta p_{n-1}^y\end{bmatrix}.$$

Further define $L_{m,n}$, $L_{m,n}^f$ by:

$$L_{m,n} = \begin{bmatrix}G_{m,n}^2\Delta p_0^x & G_{m,n}^2\Delta p_0^y\\ G_{m,n}^{m-2}\Delta p_{n-1}^x & G_{m,n}^{m-2}\Delta p_{n-1}^y\end{bmatrix}, \quad L_{m,n}^f = \begin{bmatrix}G_{m,n}^{c;2}\Delta p_0^x & G_{m,n}^{c;2}\Delta p_0^y\\ G_{m,n}^{c;m-2}\Delta p_{n-1}^x & G_{m,n}^{c;m-2}\Delta p_{n-1}^y\end{bmatrix},$$

where $G_{m,n}^q, G_{m,m}^{q;c}$, *and* $G_{m,n}^{q;f}$ are defined in (17). The matrices C, $L_{m,n}$, $L_{m,m}^c$, and $L_{m,m}^f$ are obtained from (15) and (16). The coordinate form of the expansion of (14) becomes:

$$G_{m,m}^F R_m^F = G_{m,n}^{PC} P_n^C - G_{m,m}^C R_m^C, \quad (20)$$

Where

$$G_{m,n}^{PC} = \begin{bmatrix}G_{m;n}^{P^+}\\ L_{m;n}\end{bmatrix}, G_{m,m}^C = \begin{bmatrix}G_{m;m}^{c^+}\\ L_{m;m}^f\end{bmatrix}, G_{m,m}^F = \begin{bmatrix}G_{m;m}^{f^+} & \frac{n}{m(m-1)} L_{m;m}^f\\ \left(L_{m;m}^f\right)^t & \frac{n}{m(m-1)} C\end{bmatrix},$$

From (20) we can find our unknowns as:

$$R_m^F = (G_{m,m}^F)^{-1}\left(G_{m,n}^{PC} P_n^C - G_{m,m}^C R_m^C\right). \quad (21)$$

6. Examples and comparisons

In this section, four examples are given to illustrate the effectiveness of the proposed method. The examples demonstrate the great benefits of using weighted G2-degree reduction.

Example 1: Given the Bézier curve (spiral) $P_n(t)$ of degree 19 with the control points, see Fig. 11 in (Rababah and Mann, 2013):
$P_0 = (37, 38)$, $P_1 = (43, 37)$, $P_2 = (39, 27)$, $P_3 = (29, 26)$, $P_4 = (23, 36)$,
$P_5 = (26, 50)$, $P_6 = (45, 56)$, $P_7 = (58, 47)$, $P_8 = (58, 29)$, $P_9 = (46, 14)$,
$P_{10} = (26, 6)$, $P_{11} = (5, 15)$, $P_{12} = (0, 40)$, $P_{13} = (3, 58)$, $P_{14} = (24, 68)$,
$P_{15} = (50, 75)$, $P_{16} = (79, 67)$, $P_{17} = (79, 36)$, $P_{18} = (65, 12)$, $P_{19} = (50, 0)$.

$P_n(t)$ is reduced to Bézier curve $R_m(t)$ of degree 8. Fig. 1 shows the curves with control polygons; original curve (dashed-Black); weighted G^2 (dashed-Red). Fig. 2 shows the error plots for weighted G^2-degree reduction in Example 1.

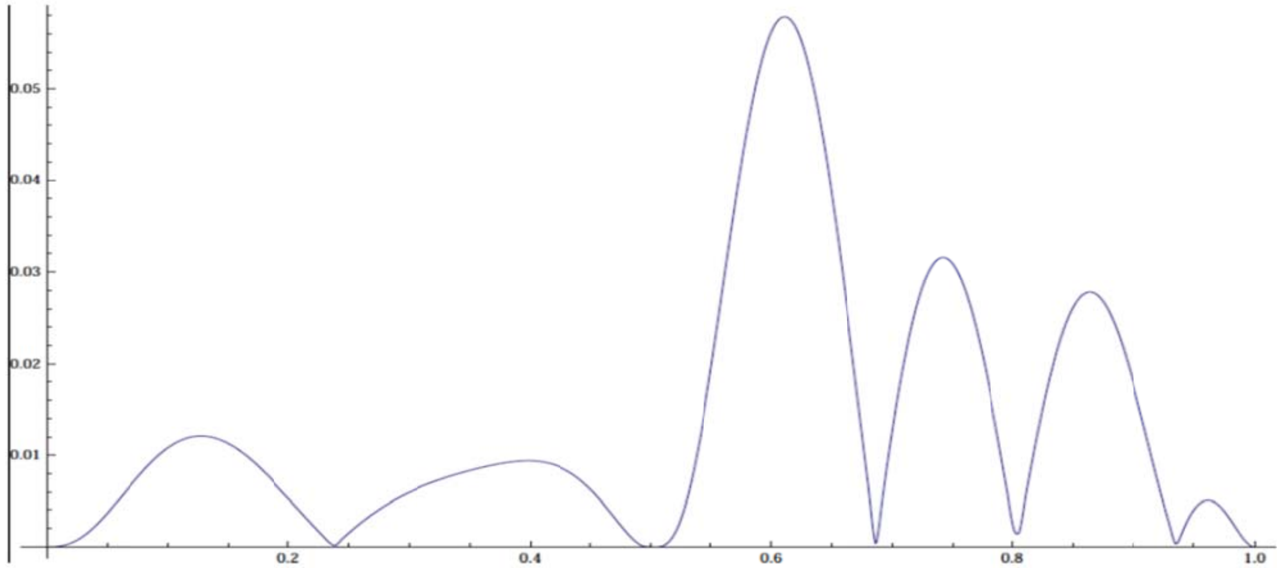

Fig. 1: The curves with control polygons

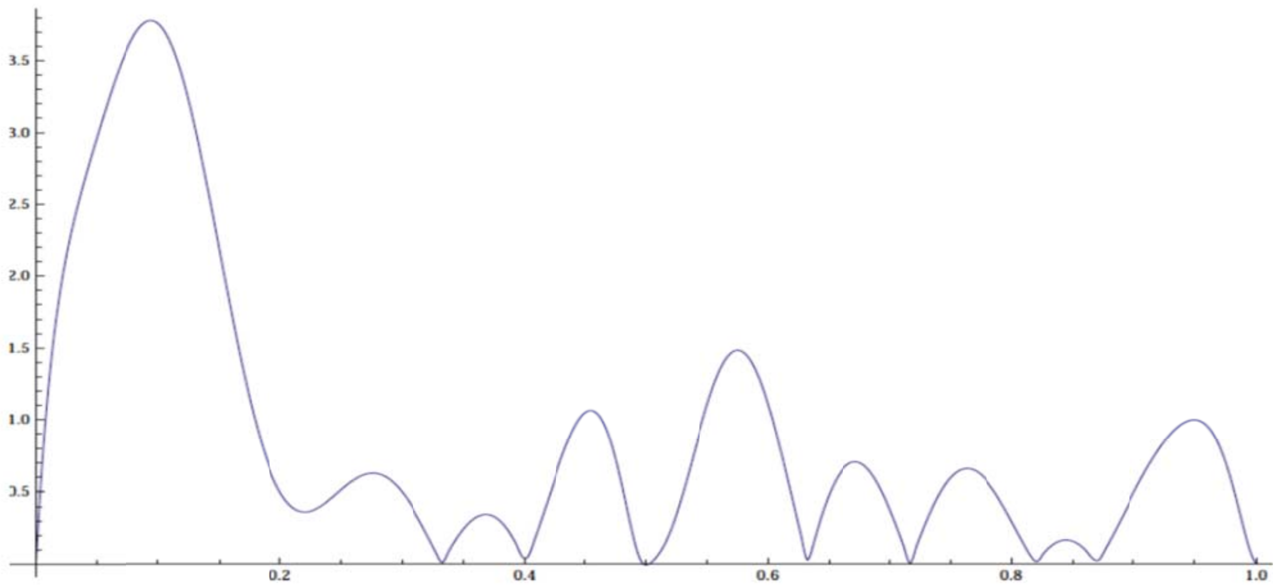

Fig. 2: The error plots for weighted G^2-degree reduction in Example 1

Example 2: Given the Bézier curve $P_n(t)$ of degree 10 with the control points, see (Lu and Wang, 2006):
$P_0 = (0, 1.2)$ $P_1 = (0.04, 0.6)$ $P_2 = (0.15473790322581, 0.507)$,
$P_3 = (0.32207661290323, 0.878)$, $P_4 = (0.30897177419355, 0.086)$,
$P_5 = (0.51864919354839, 0)$, $P_6 = (0.62449596774194, 0.8)$, $P_7 = (0.89, 0.874)$,
$P_8 = (0.92, 0.6)$, $P_9 = (0.92, 0.3)$, $P_{10} = (0.75352822580645, 0)$.

This curve is reduced to Bézier curve $R_m(t)$ of degree 6. Weighted G^2 degree reduction method is used to reduce the degree of $P_n(t)$. The original curve (Solid-Blue) and the corresponding degree reduced Bézier curve using G^2 (Solid-Red) are depicted in Fig. 3. The resulting error plot is depicted in Fig. 4.

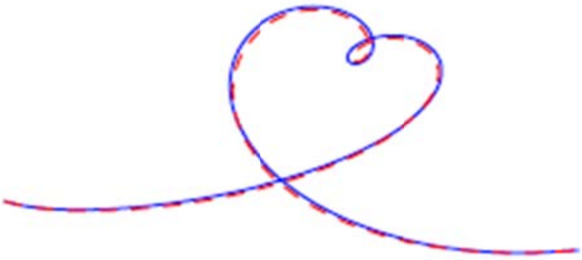

Fig. 3: The corresponding degree reduced Bézier curve using G2 (Solid-Red)

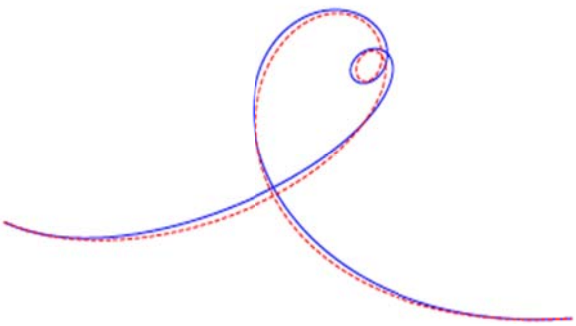

Fig. 4: The resulting error plot

Example 3: Given the Bézier curve $P_n(t)$ of degree 13 with double loop control points, see (Rababah and Mann 2013):

$P_0 = (4,9)$, $P_1 = (23,2)$, $P_2 = (49,19)$, $P_3 = (67,20)$, $P_4 = (52,48)$,
$P_5 = (0\ 23)$, $P_6 = (26,0)$, $P_7 = (71,4)$, $P_8 = (71,37)$, $P_9 = (30,54)$,
$P_{10} = (4,25)$, $P_{11} = (24,5)$, $P_{12} = (41,0)$, $P_{13} = (62,1)$

This double loop curve is reduced to Bézier curve $R_m(t)$ of degree 8. The original curve (Solid-Blue) and the corresponding degree reduced Bézier curve using G2 (Dotted-Red) are depicted in Fig. 5.

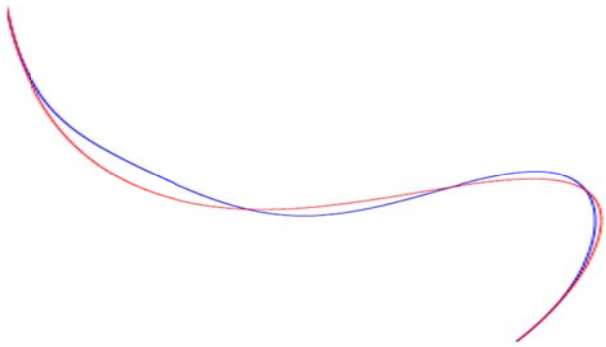

Fig. 5: The corresponding degree reduced Bézier curve using G2 (Dotted-Red)

Example 4: This example focuses on a "heart" data set, given a Bézier curve $P_n(t)$ of degree 13 with control points; see (Rababah and Mann, 2013).

$P_0 = (22,10)$, $P_1 = (37,5)$, $P_2 = (86,18)$, $P_3 = (81,23)$, $P_4 = (69,56)$,
$P_5 = (14,26)$, $P_6 = (40,3)$, $P_7 = (85,7)$, $P_8 = (85,40)$, $P_9 = (44,57)$,
$P_{10} = (18,29)$, $P_{11} = (38,9)$, $P_{12} = (55,3)$, $P_{13} = (77,5)$

The heart is reduced to Bézier curve $R_m(t)$ of degree 8. This curve is reduced to Bézier curve $R_m(t)$ of degree 8. The original curve (Solid-Blue) and the corresponding degree reduced Bézier curve using G2 (Dotted-Red) are depicted in Fig. 6.

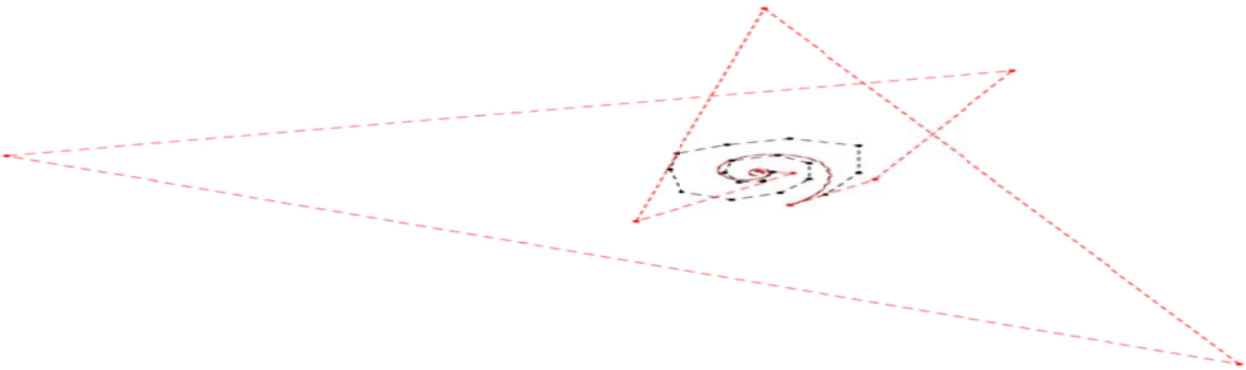

Fig. 6: The original curve (Solid-Blue) and the corresponding degree reduced Bézier curve using G2 (Dotted-Red)

7. Conclusions

In this paper, we have presented a method of weighted G2-degree reduction. Referring to the examples in Figs. 1-6, the weighted G2-degree reduction gives better results and provides less error than existing methods, see method in (Rababah and Mann, 2013) and the references therein.

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