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Stein estimation in the Conway-Maxwell Poisson model with correlated regressors

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ABSTRACT

The Poisson regression model (PRM) is widely used for count data, applicable when the response variable follows a Poisson distribution with equal dispersion. The Conway-Maxwell Poisson regression model (COMPRM) is more flexible and can handle both under-dispersion and over-dispersion. However, the COMPRM may involve correlated regressors, leading to multicollinearity, which makes the maximum likelihood estimator (MLE) inefficient. Biased estimation methods can address multicollinearity in data. This study proposes a Stein estimator, a biased estimation method, for the COMPRM that can simultaneously address correlated regressors and dispersion issues. The estimated mean square error (EMSE) is used to evaluate performance. The proposed estimator's performance is assessed both theoretically and numerically. The numerical evaluations include a simulation study under various parametric conditions and a real-world application. The results from both the simulation study and the real application demonstrate that the Stein estimator outperforms the MLE.

1. Introduction

Count data models are widely used for counting responses. One of the most popular models is the Poisson model. The Poisson model works with a single parameter as the mean and variance are equal. This property reduces its application and is unable to explain the dispersion problem. Different models are proposed to handle dispersion. Examples include the negative binomial model (Hilbe, 2011), the bell regression model (Majid et al., 2022), and the Poisson mixture, which are used for over-dispersed data and cannot handle underdispersion cases (McLachlan, 1997). A Conway-Maxwell Poisson regression model (COMPRM) can capture both over and under-dispersion for modeling queuing systems with state-dependent service rates introduced by Conwaay and Maxwell (1962).

Moreover, the count data models may be with correlated regressors. In this situation, the maximum likelihood estimator (MLE) provides an inefficient regression coefficient estimate. To address the issue of correlated regressors, Stein proposed the Stein estimator (SE) for the LRM (Stein, 1960). Stein estimator is proposed for the logistic regression model. In his study, he presented three biased estimators, ridge, Stein, and principal component regression, and compared them using a Monte Carlo simulation study, but no theoretical comparison is provided (Schaefer, 1986). For the Inverse Gaussian Regression Model, SE performs as an efficient proposed estimator (Akram et al., 2021).

The SE for the logistic regression was proposed by Schaefer (1986). The SE for the Poisson regression model as a case of count data model deals with correlated regressors and equal dispersion by Amin et al. (2022). Akram et al. (2021) considered the SE for the inverse Gaussian regression model. Recently, the SE was adapted for the beta regression model (Amin et al., 2023a). The response variable might be in count form with overdispersion, underdispersion, and correlated regressors. Therefore, we need a biased estimator for the COMPRM. We are considering using the SE as a biased estimator to address the issue of correlated regressors in the COMPRM. Although some researchers have studied this biased estimator, there is no study specifically on using the SE for the COMPRM to address correlated regressors. Thus, we propose a new estimator, the COMP Stein Estimator (COMPSE), to reduce the impact of correlated regressors. The remainder of this study is organized

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as follows: Section 2 introduces the COMPRM model and its estimation methods. Additionally, we explain COMPRM and COMPSE along with their MSEs and provide a theoretical comparison of the proposed estimator with the Maximum Likelihood Estimator (MLE). In Section 3, we assess the performance of the estimator using a Monte Carlo simulation. Section 4 involves the application of the model to real data. The final section summarizes the conclusions of the study.

2. Methodology

2.1. The COMP regression model

The COMP distribution is flexible enough to handle the dispersion in count data with an additional parameter (ν) and deals with both overdispersion ($\nu < 1$) and under dispersion ($\nu > 1$). The COMP distribution is the generalization of some well-known discrete distributions. When ν =1, then the COMP distribution approaches the Poisson distribution, when ν =0 and λ< 1, the COMP distribution is converted to Geometric distribution, and when $\nu \to \infty$, the COMP distribution approaches the Bernoulli distribution with probability $\left(\frac{\lambda}{1+\lambda}\right)$. Suppose y is a random variable and follows a COMP (λ, ν) with a probability mass function as

$$P(Y=y;\lambda,\nu)=\frac{1}{Z(\lambda,\nu)}\frac{\lambda^y}{(y!)^\nu},\quad y=0,1,2,,,,\infty \tag{1}$$

where, $Z(\lambda,\nu)=\sum_{n=0}^{\infty}\frac{\lambda^n}{(n!)^\nu}$ is the normalizing constant, λ is the mean parameter, and ν ($\nu > 0$) is the dispersion parameter (Chatla and Shmueli, 2018). The mean and variance of the COMP distribution using an asymptotic expression for Z in Eq. 1 are, respectively (Shmueli et al., 2005).

$$E(Y)\approx\lambda^{\frac{1}{\nu}}+\frac{1}{2\nu}-\frac{1}{2},$$
$$Var(Y)\approx\frac{1}{\nu}\lambda^{\frac{1}{\nu}} \tag{2}$$

The reparametrized COMP function is obtained by setting $\mu=\lambda^{\frac{1}{\nu}}$ in Eq. 1 (Guikema and Goffelt, 2008). The new formulation of (1) is given as,

$$P(Y=y;\mu,\nu)=\frac{1}{S(\mu,\nu)}\left(\frac{\mu^y}{y!}\right)^\nu,\ y=0,1,2,,,,\infty \tag{3}$$

where,

$$S(\mu,\nu)=\sum_{n=0}^{\infty}\left(\frac{\mu^n}{n!}\right)^\nu \tag{4}$$

The mean and variance of the distribution from Eq. 3 are derived as $E(Y)\approx\mu+\frac{1}{2}\nu-\frac{1}{2}$ and $Var(Y)\approx\frac{\mu}{\nu}$, it becomes accurate for $\mu>10$ and $\nu\le 1$ (Shmueli et al., 2005). Now, the new parameterization allows μ and ν as centering and shape parameters, respectively. The COMPRM consists of two types of models: the mean model and the dispersion model.

The COMP regression model is a dual-link GLM, as mean and variance depend on covariates. Y is the count variable (response variable), x_i and z_i are the covariates used in the mean link function and variance link function with p and q terms, respectively (Francis et al., 2012).

$$ln(\mu_i)=\beta_0+\sum_{j=1}^{p}\beta_j x_{ij}=x_i^t\beta \tag{5}$$
$$ln(\nu_i)=\delta_0+\sum_{k=1}^{q}\delta_k z_{ik}=z_i^t\delta \tag{6}$$

The mean and variance models in Eqs. 5 and 6 are used to estimate the coefficients of the COMPRM. For simplicity, we will assume a single value of ν and use the mean model for estimation purposes. Let $\eta_i=log(\mu_i)=x_i'\beta$ is the linear predictor with a log link, where β is the vector of regression coefficients, including the intercept. Based on the new formulation, the likelihood function of Eq. 3 (Francis et al., 2012). The log-likelihood function can be written as

$$l(y_i;\beta,\nu)=\nu\sum_{i=1}^{n}y_i\eta_i-\sum_{i=1}^{n}\nu\,log(y_i!)-\sum_{i=1}^{n}log[S(\eta_i,\nu)] \tag{7}$$

For the estimation of parameters vector β and dispersion parameter υ, we solve the log-likelihood function defined in Eq. 7. For this purpose, by differentiating Eq. 7 w.r.t β and ν, it becomes (Francis et al., 2012).

$$\frac{\partial l}{\partial\beta_j}=\sum_{i=1}^{n}(y_i\nu-\frac{\partial}{\partial\eta_i}log[S(\eta_i,\nu)])\,x_{ij} \tag{8}$$
$$\frac{\partial l}{\partial\nu}=\sum_{i=1}^{n}(-log(y_i!)-\frac{\partial}{\partial\nu}log[S(\eta_i,\nu)]) \tag{9}$$

The solution to Eqs. 8 and 9 is obtained using the iterative reweighted least squares (IRLS) method (Sellers and Shmueli, 2010). To estimate the parameter β, it is necessary to fix ν, and the same procedure applies to estimate the second parameter. For more details, refer to Shmueli et al. (2005). One disadvantage of using the MLE is that the variance becomes inflated when there is severe collinearity among the explanatory variables. Under these conditions, it becomes very difficult to determine whether the regression coefficients are significant. Fixing ν, the maximum likelihood (ML) of β is,

$$\hat{\beta}_{MLE}=(S)^{-1}X^t\widehat{W}q, \tag{10}$$

where, $S=X^t\widehat{W}X$, $q=log(\hat{\mu})+\frac{(y-\hat{\mu})}{Var(\hat{\mu})}$ is a vector of the adjusted response variable, and $\widehat{W}$ is a matrix of weights, i.e. $\widehat{W}=diag(V_i)$, where $V_i=\frac{\tau_i}{\nu}+\frac{\nu_i^2-1}{24\nu_i^3}\tau_i^{-1}+\frac{\nu_i^2-1}{12\nu_i^4}\tau_i^{-2}+\frac{\nu_i^2-1}{6\nu_i^5}\tau_i^{-3}$ with $\tau_i=\frac{\hat{\mu}_i}{\nu}$. $\hat{V}$ and q both are evaluated by using the Fisher scoring procedure. The matrix MSE (MMSE) and scalar MSE of Eq. 10 are respectively given as,

$$MSE(\hat{\beta}_{MLE})=E(\hat{\beta}_{MLE}-\beta)^t(\hat{\beta}_{MLE}-\beta)$$
$$MSE(\hat{\beta}_{MLE})=\hat{\nu}tr(S)=\hat{\nu}\sum_{j=1}^{r}\frac{1}{\lambda_j}, \tag{11}$$

where, λ_j is the jth eigenvalue of the S matrix, $\hat{\nu}$ is estimated dispersion parameter.

2.2. The Stein estimator for COMP regression model

Stein (1960) proposed an estimator as a remedy for correlated regressors in the LRM. For the COMPRM, we proposed a Stein estimator to overcome the effect of correlated regressors. The proposed COMPSE is defined as:

$$\hat{\beta}_s = c\hat{\beta}_{MLE}, \tag{12}$$

where, c $(0 < c < 1)$ is the Stein parameter. When c=1, $\hat{\beta}_{COMPSE} = \hat{\beta}_{MLE}$. The estimated bias and covariance of Eq. 12 can be computed as,

$$Bias(\hat{\beta}_s) = E(\hat{\beta}_s) - \beta$$
$$Bias(\hat{\beta}_s) = c\beta - \beta$$
$$\text{Bias} = Bias(\hat{\beta}_s) = (c - I_r)\beta, \tag{13}$$

where, I_r is the identity matrix of order $r \times r$. The variance of the COMPSE is calculated as,

$$Cov(\hat{\beta}_s) = c^2 Cov(\hat{\beta}_{MLE})$$
$$MSE(\hat{\beta}_s) = Cov(\hat{\beta}_s) + Bias(\hat{\beta}_s)^2$$
$$MSE(\hat{\beta}_s) = \hat{v}\sum_{j=1}^{r}\frac{c^2}{\lambda_j} + (c-1)^2\sum_{j=1}^{r}\alpha_j^2, \tag{14}$$

where, α_j^2 is the jth element of $Q^t\hat{\beta}_{MLE}$ and Q is the eigenvector of the matrix $Q(\Lambda)Q^t$, where $\Lambda = diag(\lambda_j)$ and c is a biasing parameter of the COMPSE and λ_j are eigenvalues. The MSE of the COMPSE depends on the value of c. An appropriate value of the c yields the minimum MSE of the COMPSE. Therefore, we suggest some new estimating methods to estimate the value of c for the COMPSE in the next subsection.

2.3. Proposed biasing parameters

For the selection of the biasing parameter, an optimum value of the biasing parameter can be obtained by taking a derivative of Eq. 14 and equating it to zero,

$$\frac{\partial(MSE(\hat{\beta}_s))}{\partial c} = \hat{v}\sum_{j=1}^{r}\frac{2c}{\lambda_j} + 2(c-1)\sum_{j=1}^{r}\alpha_j^2 = 0$$
$$\sum_{j=1}^{r}2c(\hat{v} + \lambda_j\alpha_j^2) = 2\sum_{j=1}^{r}\alpha_j^2\lambda_j.$$

On simplification, we obtain the value of c as,

$$c = \frac{\sum_{j=1}^{r}\alpha_j^2\lambda_j}{\sum_{j=1}^{r}v + \alpha_j^2\lambda_j}. \tag{15}$$

Based on the work of Hoerl and Kennard (1970) and Kibria (2003), Eq. 15 generally can be written as

$$c_j = \frac{\alpha_j^2\lambda_j}{v + \alpha_j^2\lambda_j}$$

Furthermore, using the above expression, we proposed the following biasing parameters for the COMPSE,

$$c_1 = max(c_j), \tag{16}$$
$$c_2 = \frac{\prod_{j=1}^{r}c_j^{(1/r)}}{\max(c_j)}, \tag{17}$$
$$c_3 = \left(\frac{1}{r}\right)\sum_{j=}^{r}c_j, \tag{18}$$
$$c_4 = median(c_j), \tag{19}$$
$$c_5 = \prod_{j=1}^{r}c_j^{(1/r)}, \tag{20}$$
$$c_6 = \frac{\sum_{j=1}^{r}\alpha_j^2}{\sum_{j=1}^{r}\alpha_j^2 + v\sum_{j=1}^{r}\frac{1}{\lambda_j}}. \tag{21}$$

2.4. The theoretical comparison of the proposed estimator

The superiority of the COMPSE is compared with the MLE using the following theorem.

Lemma 2.1: Let M be a positive definite (p.d.) matrix, a be a vector of nonzero constants and c be a positive constant. Then $cM - \alpha\alpha^t > 0$ if and only if $\alpha^t M\alpha < c$ (Farebrother, 1976).

Theorem 2.1: Under COMPSE, consider c > 0, $b_{COMPSE} = Bias(\hat{\beta}_{COMPSE})$ is the bias of COMPSE then $MSE(\hat{\beta}_{MLE}) - MSE(\hat{\beta}_{COMPSE}) > 0$ if $b_{COMPSE}[\hat{v}(S)^{-1} - \hat{v}c^2((S)^{-1})]\, b_{COMPSE}^t < 1$.

Proof: The difference in MSE from Eqs. 11 and 14 can be,

$$\Delta_1 = MMSE(\hat{\beta}_{MLE}) - MMSE(\hat{\beta}_{COMPSE})$$
$$= \hat{v}[(S)^{-1} - c^2((S)^{-1})] - b_{COMPSE}b_{COMPSE}^t. \tag{22}$$

From Eq. 22, we can write it as,

$$= \hat{v}(S)^{-1}[1 - c^2] - b_{COMPSE}b_{COMPSE}^t$$

The difference between the scalar MSE functions of MLE and COMPSE is as,

$$MSE(\hat{\beta}_{MLE}) - MSE(\hat{\beta}_{COMPSE})$$
$$= \sum_{j=1}^{r}\left(\frac{\hat{v}}{\lambda_j} - \frac{\hat{v}c^2}{\lambda_j} + (c-1)^2\alpha_j^2\right)$$
$$= \sum_{j=1}^{r}\left(\hat{v}\frac{(1-c^2)}{\lambda_j} + (c-1)^2\alpha_j^2\right).$$

On simplifying the results, we get

$$MSE(\hat{\beta}_{MLE}) - MSE(\hat{\beta}_{COMPSE})$$
$$= \hat{v}\sum_{j=1}^{r}\left(\frac{(1-c^2) + \lambda_j(c-1)^2\alpha_j^2}{\lambda_j}\right).$$

The expression $\hat{v}[(S)^{-1}c^2]$ is p.d if $[1 - c^2] > 0$. Thus if $0 < c < 1$, then the theorem is completed by Lemma 2.1 and it is enough to prove that the COMPSE is superior to the MLE in the form of scalar MSE for the COMPRM.

3. Monte Carlo simulation study

This section contains a numerical evaluation of the proposed estimator and a comparison with MLE using a Monte Carlo simulation. For this purpose, various factors are taken with different values. These factors include sample size, dispersion, correlated regressors, and the number of explanatory variables. The assumed values of these factors are given in Table 1.

Table 1: Assumed values of different factors for simulation study

Factors	Notation	Values
Number of explanatory variables	p	3,6,9,12
Number of replicates	R	1000
Dispersion parameter	v	0.85,1,1.25
Sample size	n	50,100,150,200
Degree of correlation	ρ^2	0.8,0.9,0.95,0.99

The response variable of the COMPRM is generated from a $CMP\ (\mu_i, v)$ distribution, where:

$$\mu_i = exp(\beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip}),\ \ i = 1, \ldots, n. \quad (23)$$

The correlated explanatory variables are generated as follows (Kibria, 2003).

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{i(j+1)}, i = 1, \ldots, n; j = 1, \ldots, p. \quad (24)$$

where, z_{ij} are the independent standard normal pseudo-random numbers. The regression parameters are selected in such a way that $\sum_{j=1}^{p} \beta_j^2 = 1$, which is a commonly used restriction in the field (Amin et al., 2023b). For the different combinations of n, p, ρ, v, the data is repeatedly generated 1000 times. The MSE criteria is used to gauge the performance of the estimators, which is defined by,

$$MSE(\hat{\beta}) = \frac{\sum_{i=1}^{R}(\hat{\beta}_i - \beta)^t(\hat{\beta}_i - \beta)}{R}, \quad (25)$$

where, $(\hat{\beta}_i - \beta)$ is the difference between the true parameter and estimated vectors of the proposed and other considered estimators at ith replication, and R represents the number of replications.

3.1. Results and discussions

The simulation study is performed under the various factors listed in Table 1. The estimated MSEs of the considered estimators are given in Tables 2-13. The summary of simulation results is as follows,

- Table 2 presents the estimated mean square error (EMSE) for $p=3$ and v=0.85 for the overdispersion case. It is observed that COMPSE with c_5 at sample size n=50,100,150, and 200 has minimum EMSE for all levels of multicollinearity as compared to the MLE and COMPSE with all other proposed Stien parameter estimators.
- On comparing the results of the proposed estimator concerning sample size, it is observed that an increase in sample size causes a decrease in the values of EMSEs. From Table 2, it is observed that for a fixed level of multicollinearity 0.80, p=3, and v=0.85, the EMSEs are 0.8777, 0.8440, 0.8835, and 0.8484, respectively. Hence, the gradual decrease in values of EMSEs shows the efficiency of the proposed estimator to combat multicollinearity by increasing the sample size.
- From Table 3, for all levels of multicollinearity, when p=3, v=1, and n=50, the values of EMSEs of c_5 are 0.9797, 1.6971, 1.8197 and 5.8784. So, the performance as a function of multicollinearity for the fixed *n, p, and v* shows an increasing trend as the level of multicollinearity increases for the EMSE of the COMPSE. The same pattern is observed in Tables 2-13.
- Tables 2, 5, 8, and 11 present the EMSE for p=3, 6, 9, and 12, respectively, showing that as the number of explanatory variables increases, the EMSEs of the estimators also increase. For p=*3*, Tables 2, 3, and 4 represent the estimated MSE for overdispersion, equidispersion, and underdispersion, respectively. The results clearly show that the EMSE is the least affected by overdispersion as compared to the equal and under-dispersion cases.

4. Application: Plastic plywood data

In this section, the performance of the proposed estimator is evaluated with the help of a real-life dataset that is related to the plastic plywood dataset. This application was considered by many researchers with different variables (Azaman et al., 2013; Demirkir et al., 2013; Fang et al., 2014). We consider this application to evaluate the performance of our proposed method and compare it with the MLE. This application consists of n=100 observations, where the response variable y represents the number of defects that may increase or decrease per laminated plastic plywood area. The four explanatory variables include volumetric shrinkage (x_1), assembly time (x_2), wood density (x_3) and drying temperature (x_4).

The estimated dispersion parameter is found to be $\hat{v} = 0.9614$, which indicates that there is overdispersion in the data set. In the regression model, commonly used methods are variance inflation factor (VIF) and condition index (CI) to test the multicollinearity among the explanatory variables. The $CI = \sqrt{\lambda_{max}/\lambda_{min}} = 8634.73$ of this data set shows severe multicollinearity among the explanatory variables. Hence, we use the COMPSE to overcome the effect of correlated explanatory variables in the COMPRM. The MSEs of the MLE and COMPSE with different shrinkage parameters are computed using Eqs. 11 and 14, respectively. The estimated regression coefficients and MSEs of different shrinkage parameters of the MLE, COMPSE are mentioned in Table 14. On comparing the performance of the COMPRM estimators, it is observed that our newly proposed estimator (COMPSE) with all five Stein parameters outperforms as compared to the MLE. Furthermore, when there are highly correlated regressors, MLE is the estimator that is most negatively affected.

5. Conclusion

In this study, we introduced a new estimator, the COMPSE, for the COMPRM to reduce the impact of correlated regressors. We evaluated the proposed estimator using a Monte Carlo simulation study, with the EMSE as the performance criterion, where a lower EMSE indicates better performance.

Table 2: EMSEs for $\nu = 0.85$ and p=3

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	1.8731	1.8540	0.8804	0.8810	1.0033	**0.8777**	1.0032
	0.9	3.7777	3.7405	1.3522	1.6757	1.9509	**1.3453**	1.9268
	0.95	6.3936	6.3326	2.1981	2.9232	3.5155	**2.1831**	2.9709
	0.99	36.5167	36.248	9.7654	16.282	20.312	**9.6973**	14.9107
100	0.8	1.6016	1.5950	0.8449	0.8377	0.9673	**0.8440**	0.9471
	0.9	3.2445	3.2312	1.2556	1.5366	1.8255	**1.2531**	1.7311
	0.95	5.9275	5.9035	1.9772	2.7852	3.4530	**1.9718**	2.9880
	0.99	30.8726	30.7512	9.1390	15.311	19.717	**9.1046**	14.0608
150	0.8	1.7464	1.7419	0.8842	0.9127	1.0502	**0.8835**	1.0923
	0.9	2.8779	2.8707	1.2421	1.4511	1.7099	**1.2405**	1.6625
	0.95	5.6424	5.6276	1.9006	2.7152	3.4022	**1.8974**	3.0410
	0.99	24.2021	24.1431	6.2130	11.498	15.362	**6.1995**	10.9629
200	0.8	1.6339	1.6308	0.8488	0.8685	0.9873	**0.8484**	1.0202
	0.9	3.1381	3.1319	1.2212	1.5147	1.8416	**1.2200**	1.7888
	0.95	5.7624	5.7516	2.0636	2.8449	3.4706	**2.0610**	3.0242
	0.99	20.2210	18.2341	5.1223	10.2120	14.7563	**5.0100**	10.1180

Bold indicated the smaller EMSE

Table 3: EMSEs for $\nu = 1$ and p=3

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	2.5216	2.4943	0.9838	1.0589	1.1279	**0.9797**	1.2586
	0.9	4.3082	4.2689	1.7066	2.0857	2.4095	**1.6971**	2.3400
	0.95	6.1630	6.1047	1.8304	2.6973	3.3640	**1.8197**	2.7369
	0.99	32.3845	32.1564	5.9207	13.202	17.2830	**5.8784**	12.9325
100	0.8	1.5628	1.5558	0.8833	0.8280	0.9490	**0.8823**	0.9180
	0.9	3.6075	3.5937	1.4076	1.7506	2.0526	**1.4047**	2.0296
	0.95	6.3364	6.3088	1.9269	2.8399	3.5250	**1.9215**	2.9814
	0.99	31.6043	31.4819	8.6775	15.6335	20.8690	**8.6456**	14.6645
150	0.8	1.5385	1.5344	0.8700	0.8412	0.9892	**0.8695**	0.9504
	0.9	3.0156	3.0077	1.0583	1.3934	1.7623	**1.0572**	1.6466
	0.95	5.9811	5.9659	2.2116	3.0050	3.7501	**2.2077**	3.1833
	0.99	30.9721	30.8964	8.5471	15.0428	19.7190	**8.5288**	15.1740
200	0.8	1.7666	1.7632	0.9769	0.9950	1.1539	**0.9762**	1.1245
	0.9	3.0860	3.0800	1.1884	1.5256	1.9210	**1.1873**	1.7502
	0.95	5.2988	5.2891	2.0396	2.7508	3.3882	**2.0371**	2.9301
	0.99	26.1379	26.0902	7.3227	12.9054	17.1422	**7.3106**	12.8037

Bold indicated the smaller EMSE

Table 4: EMSEs for $\nu = 1.25$ and p=3

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	3.6527	3.5115	1.1557	1.3714	1.5082	**1.1383**	1.3434
	0.9	8.3539	8.0320	1.7971	2.7803	2.9743	**1.7514**	2.4708
	0.95	15.0446	14.4979	3.6317	5.6743	6.4204	**3.5207**	4.3157
	0.99	82.8496	81.0760	18.7003	32.9074	38.3932	**18.2613**	22.5578
100	0.8	3.9743	3.9003	1.2142	1.5167	1.6707	**1.2051**	1.6109
	0.9	6.6259	6.5054	1.4900	2.4166	2.7908	**1.4765**	2.2696
	0.95	13.4735	13.2394	2.7931	5.1436	6.1981	**2.7563**	3.9640
	0.99	69.8082	68.7806	10.9614	28.0143	36.7776	**10.7996**	19.0875
150	0.8	3.7132	3.6685	1.1713	1.4597	1.6050	**1.1657**	1.5676
	0.9	5.7822	5.7136	1.4829	2.0971	2.3049	**1.4741**	1.9301
	0.95	11.4546	11.3204	2.2450	4.2160	5.0250	**2.2270**	3.3182
	0.99	65.2814	64.6202	11.0595	25.9822	33.6440	**10.9467**	19.6371
200	0.8	3.5341	3.5035	1.2603	1.4582	1.5572	**1.2555**	1.5249
	0.9	6.0415	5.9888	1.7040	2.3597	2.6470	**1.6952**	1.9957
	0.95	11.8783	11.7739	2.8339	4.8364	5.9326	**2.8151**	4.0940
	0.99	61.9594	61.4578	15.6510	27.4905	34.2896	**15.5287**	18.7061

Bold indicated the smaller EMSE

Table 5: EMSEs for $\nu = 0.85$ and p=6

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	5.5244	5.4716	1.3559	1.8922	2.0588	**1.3504**	2.3479
	0.9	10.0862	9.9881	1.7698	3.0566	3.3763	**1.7598**	3.5456
	0.95	21.7299	21.4923	4.0275	7.2244	8.3026	**3.9910**	7.9582
	0.99	97.0780	96.3202	12.2767	28.0876	28.1059	**12.2013**	28.9626
100	0.8	4.4131	4.3941	1.1299	1.5226	1.7268	**1.1280**	2.049
	0.9	8.0001	7.9648	1.5735	2.66216	2.9933	**1.5696**	3.2710
	0.95	13.8590	13.7984	2.6634	4.4330	4.4197	**2.6549**	5.4104
	0.99	70.3545	70.0583	11.1222	23.169	26.6673	**11.0810**	25.5155
150	0.8	4.0898	4.0784	1.1505	1.4857	1.6306	**1.1492**	1.9053
	0.9	6.7151	6.6967	1.3993	2.2573	2.5946	**1.3974**	2.9771
	0.95	11.3812	11.3530	2.3845	4.1255	4.8588	**2.3804**	5.0517
	0.99	64.0067	63.8508	12.0997	22.4253	23.2276	**12.0724**	26.1653
200	0.8	3.6062	3.5991	1.0294	1.3149	1.4493	**1.0287**	1.7550
	0.9	6.9126	6.8991	1.6564	2.5376	2.9479	**1.6546**	3.3209
	0.95	11.5818	11.5596	2.1800	3.9879	4.2929	**2.1772**	4.8047
	0.99	61.2940	61.1754	8.4401	20.4894	24.8705	**8.4255**	25.8117

Bold indicated the smaller EMSE

Table 6: EMSEs for $\nu = 1$ and p=6

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	9.7380	9.5465	1.2930	2.2997	2.2568	**1.2812**	2.6322
	0.9	16.7992	16.4948	2.4856	4.6955	4.7940	**2.4543**	4.9610
	0.95	29.3329	28.7969	3.3275	7.3523	7.0641	**3.2781**	7.0416
	0.99	158.2569	156.1642	16.2835	40.8691	34.8033	**16.0679**	36.7931
100	0.8	5.4680	5.4195	1.0782	1.5418	1.5392	**1.0753**	1.8331
	0.9	10.8507	10.7598	1.8176	3.1737	3.4687	**1.8089**	3.5306
	0.95	19.6763	19.5097	3.1715	5.6516	5.3441	**3.1510**	5.9439
	0.99	102.6834	101.993	14.5491	31.1315	29.3552	**14.4592**	30.7586
150	0.8	4.8208	4.7946	1.1193	1.5207	1.6000	**1.1171**	1.8253
	0.9	8.2393	8.1952	1.3440	2.2222	2.1182	**1.3409**	2.5607
	0.95	18.2975	18.1992	2.7970	5.4004	5.3754	**2.7861**	5.9125
	0.99	88.0189	87.5731	12.8577	25.7513	24.5624	**12.7971**	26.6877
200	0.8	4.5940	4.5759	1.0258	1.3803	1.3820	**1.0247**	1.6945
	0.9	10.0725	10.0327	1.8709	3.0711	3.2428	**1.8663**	3.6242
	0.95	16.2784	16.2148	2.3754	4.8462	4.9285	**2.3688**	5.7353
	0.99	73.5241	73.2503	9.5098	21.1291	21.1951	**9.4776**	21.0472

Bold indicated the smaller EMSE

Table 7: EMSEs for $\nu = 1.25$ and p=6

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	12.4644	11.9755	1.6703	2.8483	2.7768	**1.6299**	3.0068
	0.9	24.1352	23.2601	2.6034	5.5108	4.7107	**2.5360**	5.0147
	0.95	43.8464	42.3519	4.0853	9.6509	7.2007	**3.9730**	8.0726
	0.99	242.5266	237.045	23.7372	57.3941	50.6290	**23.1779**	45.0071
100	0.8	8.5819	8.4198	1.2109	2.0600	1.9042	**1.2029**	2.1939
	0.9	16.8357	16.5257	2.1273	4.1330	3.8733	**2.1009**	4.0861
	0.95	29.1483	28.6094	3.0947	6.6175	5.5064	**3.0514**	5.7289
	0.99	143.819	141.6595	13.1633	34.3678	27.7705	**12.9960**	28.6092
150	0.8	7.9381	7.8411	1.2152	1.9477	1.8511	**1.2099**	2.1776
	0.9	14.2010	14.0362	1.7782	3.5553	3.5147	**1.7672**	3.7716
	0.95	27.8953	27.5761	3.3276	6.8072	5.5076	**3.2995**	6.6319
	0.99	125.6663	124.3192	10.7117	29.5072	21.8648	**10.6022**	23.5085
200	0.8	8.1053	8.0344	1.4329	2.1181	2.0163	**1.4272**	2.3321
	0.9	14.8690	14.7396	2.1142	3.8277	3.5558	**2.1021**	3.9285
	0.95	21.4289	21.2401	2.0310	4.7601	4.3053	**2.0201**	4.3892
	0.99	129.477	128.5728	10.4659	29.9874	23.5663	**10.3983**	30.4567

Bold indicated the smaller EMSE

Table 8: EMSEs for $\nu = 0.85$ and p=9

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	12.1742	12.0412	1.6200	3.0387	3.1966	**1.6104**	4.0494
	0.9	20.1100	19.8980	2.4295	4.8852	5.0000	**2.4120**	6.1762
	0.95	44.9510	44.4881	4.7080	11.5664	11.5920	**4.6714**	13.3608
	0.99	189.7279	188.2678	15.8642	45.5422	43.5135	**15.7598**	46.8489
100	0.8	7.9619	7.9266	1.3268	2.1726	2.1689	**1.3240**	2.9569
	0.9	15.27	15.1989	1.9019	3.9662	4.0940	**1.8964**	5.3999
	0.95	27.2705	27.1466	3.1270	6.6739	6.1911	**3.1166**	8.7935
	0.99	131.5003	131.0075	15.3322	35.2562	36.1952	**15.2779**	42.0009
150	0.8	8.9205	8.8954	1.4012	2.4713	2.4747	**1.3993**	3.3992
	0.9	14.1834	14.1451	2.1094	3.8912	3.9483	**2.1058**	5.2536
	0.95	30.6178	30.5325	4.4406	8.6451	8.6224	**4.4303**	11.1534
	0.99	157.0304	156.641	19.0963	42.8419	43.4421	**19.0511**	56.3899
200	0.8	7.6837	7.6684	1.3629	2.2028	2.3043	**1.3616**	3.1764
	0.9	15.4263	15.3952	2.1455	4.5035	5.0588	**2.1426**	6.1252
	0.95	28.4487	28.3901	3.8966	7.9984	8.1217	**3.8902**	10.5169
	0.99	143.018	142.7522	16.3336	39.3749	41.3574	**16.3061**	51.0804

Bold indicated the smaller EMSE

Table 9: EMSEs for $\nu = 1$ and p=9

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	18.2156	17.8366	1.6494	3.5982	3.2220	**1.6304**	4.1569
	0.9	30.5285	29.9479	2.8947	6.4574	5.8005	**2.8567**	7.1828
	0.95	57.0240	56.0273	3.8033	11.2609	9.7477	**3.7550**	12.9065
	0.99	315.518	312.7813	26.6251	69.7525	60.4392	**26.4294**	72.2782
100	0.8	11.8646	11.7589	1.3819	2.6686	2.3961	**1.3764**	3.6635
	0.9	20.4850	20.3032	2.2728	4.7265	4.3521	**2.2592**	5.5805
	0.95	37.6053	37.2942	3.2184	7.9611	7.1267	**3.1980**	9.1454
	0.99	186.9961	185.8586	14.221	38.9949	33.1379	**14.1346**	42.4321
150	0.8	12.2911	12.2252	1.6152	3.0121	3.0144	**1.6107**	3.8892
	0.9	21.7443	21.6253	2.4081	4.9657	4.5030	**2.3991**	5.9018
	0.95	43.2059	42.9662	4.2817	10.0226	9.4526	**4.2617**	11.6322
	0.99	211.1366	210.2504	16.1405	47.3655	42.7969	**16.0792**	55.2976
200	0.8	10.7101	10.6662	1.4143	2.6393	2.7466	**1.4117**	3.3266
	0.9	20.8369	20.7545	2.4984	5.1755	5.1097	**2.4916**	6.3813
	0.95	39.1081	38.9539	3.8493	9.2920	9.1065	**3.8370**	11.3496
	0.99	206.1406	205.471	19.4800	49.0255	43.9951	**19.4176**	59.9555

Bold indicated the smaller EMSE

Table 10: EMSEs for $\nu = 1.25$ and p=9

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	29.8120	28.5768	2.1102	4.9335	3.9226	**2.0495**	5.1110
	0.9	50.7526	48.7797	3.0771	8.8107	7.1843	**2.9813**	8.9824
	0.95	100.4128	97.1939	4.7484	16.3670	11.1816	**4.6287**	14.7239
	0.99	519.9196	508.1721	29.7907	94.2948	70.3630	**29.2287**	83.3667
100	0.8	16.2607	15.9571	1.3567	2.8967	2.5253	**1.3465**	3.0929
	0.9	28.8359	28.2972	2.0835	4.9465	3.8663	**2.0596**	4.9256
	0.95	62.6879	61.5431	4.0800	11.3018	8.7442	**4.0185**	11.0719
	0.99	270.8632	267.4712	14.0441	45.1085	30.6100	**13.8882**	41.0175
150	0.8	18.4796	18.2584	1.6304	3.4844	2.8335	**1.6202**	4.0118
	0.9	31.2447	30.8644	2.3371	5.6250	4.3683	**2.3175**	5.7047
	0.95	70.0785	69.2906	4.5931	13.0057	9.7575	**4.5503**	13.7462
	0.99	317.0939	314.1604	23.3084	62.8703	51.9734	**23.1080**	61.9871
200	0.8	17.3403	17.1807	1.6261	3.3796	2.9540	**1.6183**	3.9643
	0.9	35.0910	34.7739	2.6425	6.9631	6.3796	**2.6259**	7.9898
	0.95	59.5165	58.9942	4.4078	11.3089	9.3730	**4.3775**	11.2819
	0.99	309.805	307.907	19.1649	60.9199	46.9694	**19.0453**	65.999

Bold indicated the smaller EMSE

Table 11: EMSEs for $\nu = 0.85$ and p=12

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	30.2268	29.8619	2.0519	5.7249	4.8988	**2.0369**	7.4582
	0.9	55.9570	55.3580	5.0961	11.8682	11.5056	**5.0562**	14.1663
	0.95	130.3971	129.061	11.1001	30.1568	28.5111	**11.0038**	37.4655
	0.99	530.9519	527.7231	36.4428	107.1048	102.876	**36.2200**	120.1458
100	0.8	11.6471	11.5917	1.4046	2.6799	2.7015	**1.4014**	3.7111
	0.9	21.8890	21.7877	2.4383	5.0299	4.7187	**2.4306**	6.9067
	0.95	43.0677	42.8641	3.8081	9.3312	8.7583	**3.7942**	12.7959
	0.99	217.66	216.93	18.5146	50.4187	53.1009	**18.4600**	66.7717
150	0.8	11.6700	11.6357	1.6905	3.1125	3.3012	**1.6876**	4.4215
	0.9	20.1339	20.0777	2.2421	4.9004	4.9984	**2.2381**	7.0077
	0.95	41.8159	41.6944	4.1354	9.7494	9.8362	**4.1259**	14.1508
	0.99	176.3168	175.9053	17.5253	42.9961	44.5394	**17.4879**	57.3847
200	0.8	9.3094	9.2902	1.3341	2.3564	2.3524	**1.3329**	3.5667
	0.9	17.8146	17.7792	2.0370	4.5573	4.7551	**2.0345**	6.5408
	0.95	34.6227	34.5528	3.4577	8.2844	8.4306	**3.4523**	11.7178
	0.99	156.448	156.193	15.5181	38.1152	38.2063	**15.4951**	54.4191

Bold indicated the smaller EMSE

Table 12: EMSEs for $\nu = 1$ and p=12

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	48.5934	47.4954	3.3632	8.7121	8.2088	**3.3029**	10.1686
	0.9	77.4855	76.0530	4.4443	13.4171	11.6011	**4.3811**	15.7877
	0.95	150.571	147.9989	8.2944	28.4764	29.9445	**8.1640**	30.9811
	0.99	800.6394	793.2687	47.6374	145.4789	125.4188	**47.1892**	150.6116
100	0.8	16.4466	16.3000	1.6439	3.4035	3.1997	**1.6362**	4.3053
	0.9	33.5598	33.2668	2.4338	6.5862	6.0150	**2.4192**	8.5112
	0.95	62.0426	61.5142	4.5741	12.1235	10.3679	**4.5423**	13.8004
	0.99	318.6942	316.7213	17.0613	57.4023	47.2060	**16.9632**	68.0070
150	0.8	15.8929	15.8042	1.4558	3.2192	3.0713	**1.4521**	4.5957
	0.9	27.6453	27.4919	2.3008	5.3943	4.8452	**2.2922**	7.096
	0.95	55.3014	54.9955	3.9494	10.5169	8.3756	**3.9326**	13.9478
	0.99	277.5442	276.3905	20.3704	55.6302	46.0304	**20.2893**	70.2910
200	0.8	14.2363	14.1776	1.3940	2.9350	2.8181	**1.3915**	4.0668
	0.9	25.0326	24.9315	2.1893	5.0682	4.3449	**2.1836**	6.8289
	0.95	49.4963	49.2973	4.0990	10.3963	9.4561	**4.0858**	13.5885
	0.99	247.9527	247.1612	18.4930	52.1994	49.0265	**18.4388**	67.0262

Bold indicated the smaller EMSE

Table 13: EMSEs for $\nu = 1.25$ and p=12

n	ρ^2	COMPSE						
		MLE	c_1	c_2	c_3	c_4	c_5	c_6
50	0.8	66.5138	64.0073	3.0372	9.7563	7.0821	**2.9489**	10.2700
	0.9	114.975	110.9237	6.1130	18.2827	14.9224	**5.9085**	16.9588
	0.95	225.1859	219.2014	8.7937	32.8742	22.2083	**8.5763**	28.9148
	0.99	1220.882	1201.324	46.8510	180.4798	139.0976	**46.1052**	146.2542
100	0.8	23.8395	23.4027	1.4939	3.4445	2.6202	**1.4816**	3.8261
	0.9	47.6500	46.7618	2.7865	7.4517	5.9284	**2.7503**	7.9167
	0.95	93.3668	91.8145	4.5054	14.0700	10.4409	**4.4413**	14.7955
	0.99	485.9481	481.4397	17.6006	73.2696	51.4461	**17.4287**	80.1791
150	0.8	22.8557	22.5772	1.3937	3.3526	2.6201	**1.3870**	3.9435
	0.9	42.4388	41.9304	2.2662	6.5493	4.9716	**2.2484**	7.7202
	0.95	87.6691	86.7035	4.3972	13.8002	9.6368	**4.3614**	15.7909
	0.99	394.6527	391.5111	19.5328	62.4182	49.1158	**19.3725**	67.6536
200	0.8	20.6514	20.4682	1.5539	3.4525	2.8835	**1.5474**	4.3300
	0.9	38.2561	37.9073	2.3842	6.4367	5.2434	**2.3707**	7.6864
	0.95	71.4081	70.7905	3.8656	11.0292	8.2573	**3.8390**	12.2316
	0.99	358.3625	355.8587	17.0435	58.2462	45.1041	**16.9289**	62.7291

Bold indicated the smaller EMSE

Table 14: Estimated COM Poisson regression coefficients and MSEs for plastic plywood data

Terms	MLE	COMPSE					
		c_1	c_2	c_3	c_4	c_5	c_6
Constant	1.4397	1.4393	0.5670	0.8577	1.2232	0.5668	0.4532
x_1	0.4937	0.4936	0.1944	0.2941	0.4195	0.1944	0.1554
x_2	0.6011	0.6009	0.2367	0.3581	0.5107	0.2367	0.1892
x_3	0.3818	0.3817	0.1504	0.2275	0.3244	0.1503	0.1202
x_4	0.7401	0.7399	0.2915	0.4409	0.6288	0.2914	0.2330
MSE	6.7689	4.9352	3.5339	3.1553	5.3734	2.3767	2.3100

With a fixed sample size, explanatory variables, and dispersion parameter, the EMSEs of the COMPSE, using all proposed Stein parameters, were lower than those of the MLE. When the model included correlated regressors, the EMSE decreased as the sample size increased. Additionally, multicollinearity and the number of regressors directly affected the performance of the estimators. The EMSEs were lowest for overdispersion compared to underdispersion and equidispersion. Therefore, based on simulation and real application results, we conclude that our newly proposed estimator for the COMPRM is more appropriate than the MLE in the presence of multicollinearity and dispersion.

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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