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Conjugacy classes and conjugate graph of a K-metacyclic group

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ABSTRACT

The relationship between algebraic structures and graphs has become an interesting topic of research nowadays. In this paper, we have considered the conjugate graph related to the conjugacy relation of a group. The vertices of the said graph are the noncentral elements of the group, and two vertices are adjacent if they are conjugate. For this particular study, we focused on the conjugate graph of a K-metacyclic group of order $p(p-1)$. We first determine the conjugacy classes of this group and then obtain its conjugate graph. Various graph properties such as planarity, line graph, complement graph, clique number, dominating number, spectrum, and Laplacian are also studied in this paper.

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1. Introduction

The construction of graphs with the help of groups has become one of the most interesting topics of research, and many works have been done in relation to this in the past few years (Fortunato, 2010; Kumar et al., 2021). The concept of a conjugate graph was first introduced by Erfanian and Tolue (2012) and attracted many researchers afterward (Prasad et al., 2019; Alolaiyan et al., 2019; Ling and Qin, 2022). They defined it to be a graph whose vertices are the noncentral elements of a group, and two vertices are adjacent if they are conjugate. Several researchers are working to find the conjugate graph of specific finite groups. Some of these include the work by Zulkarnain et al. (2019), where they found the conjugacy classes of some finite p-groups and then determined their conjugate graph. Another work is by Alimon et al. (2017), who presented the adjacency matrix of a conjugate graph of some metacyclic 2-groups. Motivated by these works, we found the conjugate graph of a class of K-metacyclic group and then studied its different properties. Here, we have considered the class of the K-metacyclic group that is of order $p(p-1)$. Hence, when we refer to a K-metacyclic group, its order will always be taken as $p(p-1)$ throughout this paper. We first give some preliminaries on group theory and graph theory and then discuss some results on the conjugate graph of a class of K-metacyclic group.

2. Preliminaries

In this section, some preliminary definitions and results about group theory and graph theory are cited, which will be used throughout this paper.

- **Conjugate of an element:** Suppose G is a finite group. Two elements a and b of G are called conjugate if there exists g in G with $g^{-1}ag = b$ (Herstein, 2006).
- **Conjugacy class:** Conjugacy is an equivalence relation, and the equivalence class that contains the element a in G denoted by $cl(a)$ is called the conjugacy class of a (Herstein, 2006).
- **Center of a group:** Let G be a group and let $Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$. Then, $Z(G)$ is called the center of the group G (Herstein, 2006).
- **Primitive roots modulo m:** An integer b is a primitive root modulo m if b is coprime to m and the order of $b(\text{mod } m)$ is $\phi(m)$ (Childs, 2009).
- **K-metacyclic group:** It is a group of order $p(p-1)$ generated by the elements a and b with defining relations (Dutta, 1997):

$$a^p = b^{p-1} = 1;\ b^{-1}ab = a^r; (r-1, p) = 1$$

where, r is a primitive root modulo p and p is an odd prime.

- **Complete graph:** A graph G is said to be complete if every vertex in G is connected with every other vertex (Godsil and Royle, 2001).

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- **Regular graph:** A graph in which all vertices are of equal degree is called a regular graph (Godsil and Royle, 2001).
- **Girth:** The girth of a graph is the length of the shortest cycle in it (Godsil and Royle, 2001).
- **Chromatic number:** A chromatic number of a graph G, denoted as $\chi(G)$, is the minimal number of colors required to color the vertices of G in such a way that no two adjacent vertices have the same color (Erfanian and Tolue, 2012).
- **Clique number:** A subset C of vertices of Γ is called a clique if the induced subgraph on C is a complete graph. The maximum size of a clique is called the clique number of the graph G and is denoted by $\omega(G)$ (Erfanian and Tolue, 2012).
- **Independent set:** A subset X of the vertices of the graph G is called an independent set if the induced subgraph on X has no edges. The maximum size of an independent set in the graph G is called the independence number of the graph denoted by $\alpha(G)$ (Erfanian and Tolue, 2012).
- **Dominating number:** For a graph Γ and a subset S of vertices, denote by $N_\Gamma[S]$ the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$, then S is called a dominating set for Γ. The dominating number $\gamma(\Gamma)$ of Γ is the minimum size of a dominating set of the vertices of Γ (Erfanian and Tolue, 2012).
- **Planar graph:** A graph is called planar if it can be drawn without crossing edges (Godsil and Royle, 2001).
- **Line graph:** The line graph of a graph X is the graph $L(X)$ with the edges of X as its vertices, and where two edges of X are adjacent if and only if they are incident in X (Godsil and Royle, 2001).
- **Complement graph:** The complement $\bar{X}$ of a graph X has the same vertex set as X, where vertices x and y are adjacent in $\bar{X}$ if and only if they are not adjacent in X (Godsil and Royle, 2001).
- **Adjacency matrix:** The adjacency matrix$A(X)$ of a directed graph X is the integer matrix with rows and columns indexed by the vertices of X, such that the uv-entry of $A(X)$ is equal to the number of arcs from u to v (Godsil and Royle, 2001).
- **Spectrum:** The spectrum of a matrix is the list of its eigenvalues together with their multiplicities (Godsil and Royle, 2001).
- **Laplacian matrix:** The Laplacian matrix of a graph $G = (V, E)$ where V is the vertex set and E is the edge set is an $n \times n$ symmetric matrix with one row and column for each node defined by, $L = D - A$, where D is the degree matrix, which is the diagonal matrix formed from the vertex degrees and A is the adjacency matrix. The diagonal elements l_{ij} of L are therefore equal to the degree of vertex v_i and off-diagonal elements l_{ij} are -1 if the vertex v_i is adjacent to v_j and 0 otherwise (Merris, 1994).
- **Conjugate graph:** A conjugate graph is a graph whose vertices are the noncentral elements of a group G and two distinct vertices are adjacent if they are conjugate. It is denoted by Γ_G^c (Erfanian and Tolue, 2012).
- **Result 2.20:** Kuratowski's theorem: A graph is nonplanar if and only if it contains a subdivision of K_5 or $K_{3,3}$ (Koltz, 1989).

3. Results and discussion

In this section, we have obtained the conjugate graph of a K-metacyclic group and then studied some of its properties. We first calculated the conjugacy classes for different values of p, taking $p = 3$, 5, and 7, and obtained their respective conjugate graphs as follows.

Taking $p = 3$, order = 3(3-1) = 6, the primitive root mod 3 is 2. Hence, the defining relation becomes as follows:

$$a^3 = b^2 = 1; b^{-1}ab = a^2$$

The elements are= $\{1, a, a^2, b, ba, ba^2\}$. The conjugacy classes are found to be as follows, and the conjugate graph is shown in Fig. 1.

C.C (1) = $\{a, a^2\}$
C.C (2) = $\{b, ba, ba^2\}$

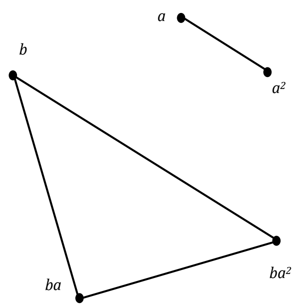

Fig. 1: Conjugate graph with $p = 3$

Taking $p = 5$, order = 5(5-1) = 20, the primitive root mod 5 is 2, 3. Hence, the defining relation becomes as follows:

$$a^5 = b^4 = 1; b^{-1}ab = a^2 \text{ (taking } r = 2)$$

The elements are:

$$=\{1, a, a^2, a^3, a^4, b, b^2, b^3, ba, ba^2, ba^3, ba^4, b^2, b^2a, b^2a^2, b^2a^3, b^2a^4, b^3a, b^3a^2, b^3a^3, b^3a^4\}$$

The conjugacy classes are found to be as follows, and the conjugate graph is shown in Fig. 2.

C.C (1) = $\{a, a^2, a^3, a^4\}$
C.C (2) = $\{b, ba, ba^2, ba^3, ba^4\}$
C.C (3) = $\{b^2, b^2a, b^2a^2, b^2a^3, b^2a^4\}$
C.C (4) = $\{b^3, b^3a, b^3a^2, b^3a^3, b^3a^4\}$

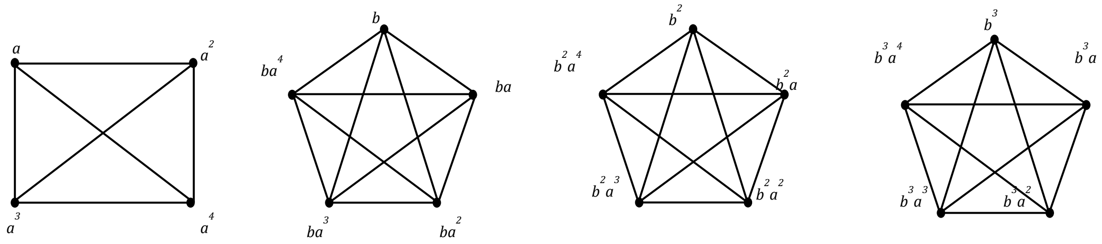

Fig. 2: Conjugate graph with $p = 5$

Taking $p = 7$, order = 7(7-1) = 42, the primitive root mod 7 is 3, 5. Hence, the defining relation becomes as follows:

$a^7 = b^6 = 1; b^{-1}ab = a^3$ (taking $r = 3$)

The elements are:

$$\{1, a, a^2, a^3, a^4, a^5, a^6, b, b^2, b^3, b^4, b^5, ba, ba^2, ba^3, ba^4, ba^5, ba^6, b^2a, b^2a^2, b^2a^3, b^2a^4, b^2a^5, b^2a^6, b^3a, b^3a^2, b^3a^3, b^3a^4, b^3a^5, b^3a^6, b^4a, b^4a^2, b^4a^3, b^4a^4, b^4a^5, b^4a^6, b^5a, b^5a^2, b^5a^3, b^5a^4, b^5a^5, b^5a^6\}$$

The conjugacy classes are found to be as follows, and the conjugate graph is shown in Fig. 3.

C.C (1) = $\{ a, a^2, a^3, a^4, a^5, a^6\}$
C.C (2) = $\{b, ba, ba^2, ba^3, ba^4, ba^5, ba^6\}$
C.C (3) = $\{b^2, b^2a, b^2a^2, b^2a^3, b^2a^4, b^2a^5, b^2a^6\}$
C.C (4) = $\{ b^3, b^3a, b^3a^2, b^3a^3, b^3a^4, b^3a^5, b^3a^6\}$
C.C (5) = $\{ b^4, b^4a, b^4a^2, b^4a^3, b^4a^4, b^4a^5, b^4a^6\}$
C.C (6) = $\{ b^5, b^5a, b^5a^2, b^5a^3, b^5a^4, b^5a^5, b^5a^6\}$

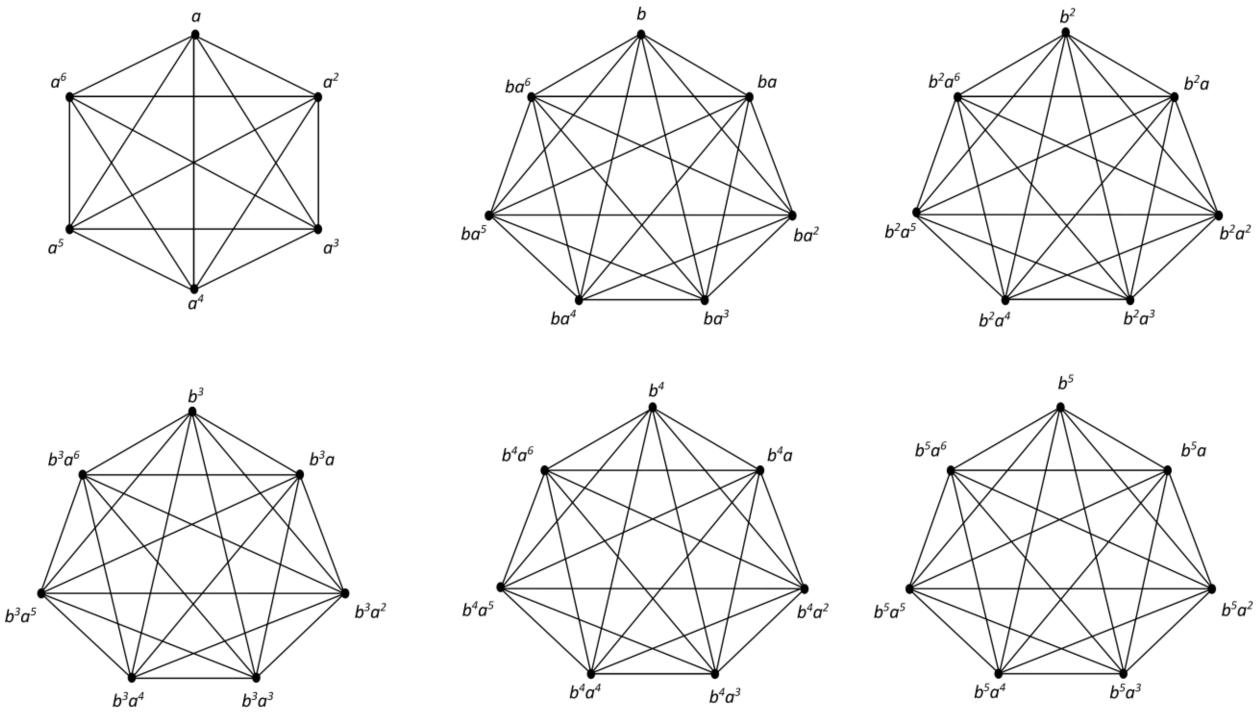

Fig. 3: Conjugate graph with $p = 7$

Continuing in this manner, if we consider the conjugacy class of a K-metacyclic group for any p, we get the following result:

Theorem 3.1: There are $p - 1$ noncentral conjugacy classes in a K-metacyclic group of order $p(p-1)$ and they are:

C.C (1) = $\{ a, a^2, a^3, \ldots, a^{p-1}\}$
C.C (2) = $\{b, ba, ba^2, ba^3, \ldots, ba^{p-1}\}$
C.C (3) = $\{b^2, b^2a, b^2a^2, b^2a^3, \ldots, b^2a^{p-1}\}$
⋮
C.C $(p-1)$ = $\{b^{p-2}, b^{p-2}a, b^{p-2}a^2, b^{p-2}a^3, \ldots, b^{p-2}a^{p-1}\}$

Proof: Let us consider a K-metacyclic group G with generators a and b. Then, the elements of this group are of the form (Hall, 2018):

$$1, a, a^2, \ldots, a^{p-1}, b, b^2, \ldots, b^{p-2}, ba, ba^2, \ldots, ba^{p-1}, b^2a, b^2a^2, \ldots, b^2a^{p-1}, \ldots, b^{p-2}a, b^{p-2}a^2, \ldots, b^{p-2}a^{p-1}.$$

Thus, we see that the elements of this group are always either the identity or of the form a^x or b^y or $b^y a^x$, where, $1 \leq x \leq p$ and $1 \leq y \leq p-1$.

For the conjugacy class of a:
Case 1: If $g \in G$ is either equal to 1 or is a power of a, then the value of $g^{-1}ag$ will be a itself as g^{-1} and g would cancel, leaving us with simply a power of a.
Case 2: If g is a power of b, say

$g = b^x \ (1 \leq x \leq p-1)$.

Then, $g^{-1} = (b^x)^{-1} = b^{p-1-x}$. Thus,

$$g^{-1}ag = (b^{p-1-x})ab^x$$
$$g^{-1}ag = b^{p-1-x}ba^r b^{x-1}$$
$$g^{-1}ag = b^{p-x}a^r b^{x-1} \quad (1)$$
$$g^{-1}ag = b^{p-x}a^{r-1}ba^r b^{x-2}$$
$$g^{-1}ag = b^{p-x}a^{r-2}ba^r a^r b^{x-2}$$
$$g^{-1}ag = b^{p-x}a^{r-3}ba^r a^{2r} b^{x-2}$$
$$g^{-1}ag = b^{p-x}a^{r-4}ba^r a^{3r} b^{x-2}$$
$$g^{-1}ag = b^{p-x}a^{r-5}ba^r a^{4r} b^{x-2}$$
$$g^{-1}ag = b^{p-x}a^{r-5}ba^{5r} b^{x-2}$$
$$\vdots$$
and so on.

This will continue until the power of a in the left of the R.H.S comes to zero. Hence, the steps will continue for r times and the expression after following the steps for r times will be as follows:

$$g^{-1}ag = b^{p-x}a^{r-r}ba^{r.r}b^{x-2}$$
$$g^{-1}ag = b^{p-x+1}a^{r^2}b^{x-2} \quad (2)$$
$$g^{-1}ag = b^{p-x+1}a^{r^2-1}ba^r b^{x-3}$$
$$g^{-1}ag = b^{p-x+1}a^{r^2-2}ba^r a^r b^{x-3}$$
$$g^{-1}ag = b^{p-x+1}a^{r^2-3}ba^r a^{2r} b^{x-3}$$
$$g^{-1}ag = b^{p-x+1}a^{r^2-4}ba^r a^{3r} b^{x-3}$$
$$g^{-1}ag = b^{p-x+1}a^{r^2-4}ba^{4r} b^{x-3}$$
$$\vdots$$
and so on.

This will continue until the power of a in the left side of the R.H.S comes to zero, and hence the steps must continue for r^2 times, and the expression after all the steps will be as follows:

$$g^{-1}ag = b^{p-x+1}a^{r^2-r^2}ba^{r^2 r}b^{x-3}$$
$$g^{-1}ag = b^{p-x+2}a^{r^3}b^{x-3} \quad (3)$$
$$\vdots$$
and so on.

Now, looking at Eqs. 1, 2, and 3, if we continue to follow the same steps and keep reducing the power of b in the right side of the R.H.S until it comes to zero, the end expression will be:

$$g^{-1}ag = b^{p-x+(x-1)}a^{r^x}b^{x-x}$$
$$g^{-1}ag = b^{p-1}a^{r^x}$$
$$g^{-1}ag = a^{r^x}$$

This will give distinct powers of a from 1 up to $p-1$ as $1 \leq x \leq p-1$ and $a^p = 1$. Hence, the conjugacy class of a contains all the powers of a.

Case 3: If g is of the form $b^y a^x$, where $1 \leq x \leq p$ and $1 \leq y \leq p-1$, then,

$$g^{-1} = (b^y a^x)^{-1} = a^{-x}b^{-y} = a^{p-x}b^{p-y-1}$$

and,

$$g^{-1}ag = (a^{p-x}b^{p-y-1})a(b^y a^x)$$
$$g^{-1}ag = a^{p-x}(b^{p-y-1}ab^y)a^x$$
$$g^{-1}ag = a^{p-x}a^{r^y}a^x \text{ (from case 2)}$$
$$g^{-1}ag = a^{p+r^y}$$
$$g^{-1}ag = a^{r^y}$$

which is again a power of a. Hence, the conjugacy class of a contains exactly all the powers of a. Meaning the conjugacy class of $a = \{a, a^2, a^3, \ldots, a^{p-1}\}$

For the conjugacy class of b:
Case 1: If $g = 1$ or g is a power of b, then the value of $g^{-1}bg$ will be b itself as g^{-1} and g would cancel each other.
Case 2: If g is a power of a, say

$g = a^x (1 \leq x \leq p)$

then,

$$g^{-1} = a^{-x} = a^{p-x}$$

and,

$$g^{-1}bg = a^{p-x}ba^x$$
$$g^{-1}bg = a^{p-x-1}ba^r a^x$$
$$g^{-1}bg = a^{p-x-2}ba^r a^r a^x$$
$$g^{-1}bg = a^{p-x-3}ba^r a^{2r} a^x$$
$$g^{-1}bg = a^{p-x-4}ba^r a^{3r} a^x$$
$$g^{-1}bg = a^{p-x-5}ba^r a^{4r} a^x$$
$$g^{-1}bg = a^{p-x-2}ba^{5r} a^x$$
$$\vdots$$
and so on.

This will continue until the power of a in the left side reduces to zero, and hence the steps will go on for $p-x$ times, and the end expression after those $p-x$ steps will be as follows:

$$g^{-1}bg = a^{p-x-(p-x)}ba^{(p-x)r}a^x$$
$$g^{-1}bg = ba^{pr-xr}a^x$$
$$g^{-1}bg = ba^{x(1-r)+r}$$

The power of a in this expression will give us distinct values from 1 up to $p-1$ for different values of x and fixed p and r as $\gcd(r,p) = 1$, $1 \leq x \leq p$ and $a^p = 1$.

Case 3: If g is of the form $b^y a^x$, where $1 \leq x \leq p$ and $1 \leq y \leq p-1$

then,

$$g^{-1} = a^{-x}b^{-y} = a^{p-x}b^{p-y-1}$$

and,

$g^{-1}bg = (a^{p-x}b^{p-y-1})b(b^y a^x)$
$g^{-1}bg = a^{p-x}b^p a^x$
$g^{-1}bg = a^{p-x}ba^x$

which is of the same form as in case 2 and hence the conjugacy class of $b = \{b, ba, ba^2, ba^3, \ldots, ba^{p-1}\}$

For conjugacy class of any power of b:

For finding the conjugacy class of any power of b, say b^z, where $1 \leq z \leq p-1$, we follow the same steps as we did in finding the conjugacy class of b and simply replace b with b^z.

Thus, the conjugacy class of b^z is given by $\{b^z, b^z a, b^z a^2, \ldots, b^z a^{p-1}\}$ for each z. Hence, we conclude that there are $p-1$ conjugacy classes in total, and they are:

C.C (1) = $\{a, a^2, a^3, \ldots, a^{p-1}\}$
C.C (2) = $\{b, ba, ba^2, ba^3, \ldots, ba^{p-1}\}$
C.C (3) = $\{b^2, b^2a, b^2a^2, b^2a^3, \ldots, b^2a^{p-1}\}$
⋮
C.C $(p-1)$ = $\{b^{p-2}, b^{p-2}a, b^{p-2}a^2, b^{p-2}a^3, \ldots, b^{p-2}a^{p-1}\}$

Corollary 3.1: The center of a K-metacyclic group contains only the identity element.

Proof: Total number of elements in all conjugacy classes as observed in theorem 3.1.

$= (p-1) + p(p-2)$
$= p(p-1) - 1$
= order of the group -1

which implies that the center contains only one element, and hence, we can conclude that the center contains only the identity.

Theorem 3.2: The different values of r (the primitive root mod p) in a K-metacyclic group give the same conjugacy classes, and hence, only one conjugate graph is formed with each different r for a fixed p upto isomorphism.

Proof: Let us suppose that there exist two values of r, say x and y.

then,

$ab = ba^x$ and $ab = ba^y$
$ba^x = ba^y$
$a^x = a^y$

which implies that they perform the same operation and, hence, will give the same conjugacy classes. Similarly, if there exist t values of r, say $x_1, x_2, \ldots, x_t$,

then,

$a^{x_1} = a^{x_2} = a^{x_3} = \cdots = a^{x_t}$

hence the theorem.

Theorem 3.3: The conjugate graph of a K-metacyclic group G is a union of $p-1$ complete graphs, $p-2$ of which is of order p and the other one is of order $p-1$. This graph contains a total of $\frac{(p+1)(p-1)(p-2)}{2}$ edges.

Proof: The first part of the theorem is evident by looking at Theorem 3.1 and the definition of a conjugate graph.

For the number of edges, since the center of a K-metacyclic group only contains the identity (by corollary), hence the total number of vertices of G will be $p(p-1)-1$, and the total number of edges will be:

$$= \sum_{i=1}^{p-1} \binom{|x_i^G|}{2}$$

where $|x_i^G|$ is the size of the conjugacy class of x_i

$$= \binom{p-1}{2} + \binom{p}{2} + \binom{p}{2} + \cdots ((p-2)\text{times})$$
$$= \frac{(p-1)(p-2)}{2} + (p-2)\frac{p(p-1)}{2}$$
$$= \frac{(p-1)(p+1)(p-2)}{2}$$

Theorem 3.4: The clique number and chromatic number of the conjugate graph of a K-metacyclic group G are equal and is equal to p.

$$\omega(\Gamma_G^c) = \chi(\Gamma_G^c) = p$$

Proof: We know that for a conjugate graph, the clique number is equal to the chromatic number (Erfanian and Tolue, 2012). Since the conjugate graph of a K-metacyclic group is a union of $p(p-1)$ complete graphs and the largest of them is of size p (from Theorem 3.3), hence the clique number equals chromatic number equals p.

Example: Let X be a K-metacyclic group taking $p = 3$.Then, r(primitive root mod 3)=2 and the elements of $X = \{1, a, a^2, b, ba, ba^2\}$. The conjugate graph of X is given in the Fig. 4.

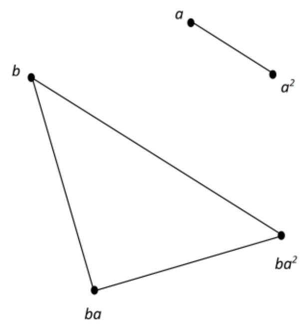

Fig. 4: Conjugate graph with $p = 3$

Clearly, Clique number $\omega(\Gamma_G^c)$ =Chromatic number $\chi(\Gamma_G^c) = 3$

Theorem 3.5: The conjugate graph of a K-metacyclic group is planar if $p = 3$ and nonplanar if $p \geq 5$.

Proof: Kuratowski's theorem states that a graph is nonplanar if and only if it contains a subdivision of K_5 or $K_{3,3}$ (Koltz, 1989).

Now, by definition, a subset C of vertices of G is called a clique if the induced subgraph on C is a complete graph, and the maximum size of a clique is called the clique number of the graph G. If a graph has a clique number that is greater than or equal to 5, then the graph clearly contains a subdivision of K_5 and hence is nonplanar. Since for the conjugate graph of a K-metacyclic group, the clique number is equal to p, where $p(p-1)$ is the order of the K-metacyclic group, we can conclude that the graph is nonplanar when $p=5$.

Now, when $p=3$, the conjugate graph is as given in the previous example, which is clearly planar, hence the theorem.

Theorem 3.6: The independence number and the dominating number of the conjugate graph of a K-metacyclic group are equal to $p-1$.

$\alpha(\Gamma_G^c) = \gamma(\Gamma_G^c) = p-1$

Proof: We know that for a conjugate graph, the independence number is equal to the dominating number (Erfanian and Tolue, 2012). Since the conjugate graph of a K-metacyclic group is a union of $p-1$ complete graphs (from Theorem 3.3), hence the independence number equals the dominating number equals $p-1$.

Example: Let X be a K-metacyclic group taking $p=5$. Then, r(primitive root mod 3)=2,3 and the elements of

$X=\{1,a,a^2,a^3,a^4,b,b^2,b^3,ba,ba^2,ba^3,ba^4,$
$b^2,b^2a,b^2a^2,b^2a^3,b^2a^4,b^3a,b^3a^2,b^3a^3,b^3a^4\}.$

The conjugate graph of X is given in the Fig. 5. Clearly, independence number $\alpha(\Gamma_G^c)=$ dominating number $\gamma(\Gamma_G^c)=4$.

Theorem 3.7: The line graph of the conjugate graph of a K-metacyclic group of order $p(p-1)$ is a disjoint union of $p-1$ regular graphs, $(p-2)$ of which are $2(p-2)$-regular graphs of order $\binom{p}{2}$ and one is a $2(p-3)$-regular graph of order $\binom{p-1}{2}$.

Proof: We first state and prove two claims:

Claim 1: The line graph formed with a complete graph of order $p-1$ will give a $(2p-3)$-regular graph of order $\binom{p-1}{2}$.

Proof: Since the number of vertices of the complete graph $=p-1$, the number of edges $=\binom{p-1}{2}$, thus, the number of vertices of the line graph $=\binom{p-1}{2}$.

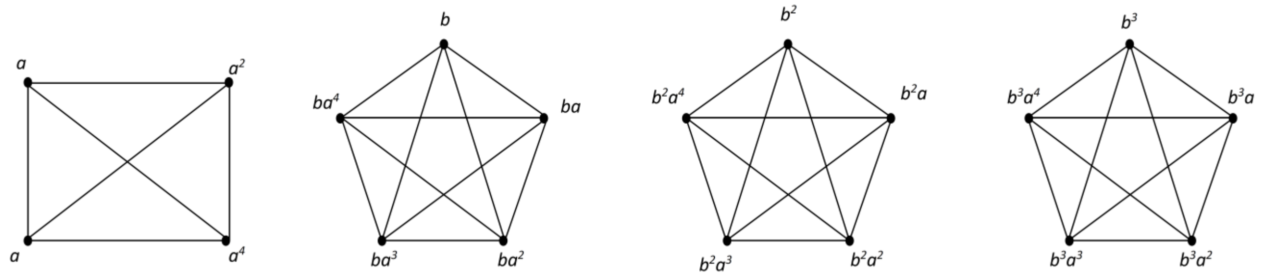

Fig. 5: Conjugate graph with $p=5$

Now, in the complete graph, the degree of each vertex is $p-1-1=p-2$. Hence, if we consider an edge and count all edges incident to that particular edge, then we see that it is equal to $p-2+p-2-1-1=2(p-3)$. Thus, the line graph formed with a complete graph of order $(p-1)$ is a $2(p-2)$-regular graph of order $\binom{p-1}{2}$.

Claim 2: The line graph formed with a complete graph of order p is a $2(p-2)$-regular graph of order $\binom{p}{2}$.

Proof: Number of vertices of complete graph$=p$, number of edges $=\binom{p}{2}$, number of vertices of line graph $=\binom{p}{2}$, in a complete graph, the degree of each vertex $=p-1$. Hence, if we consider an edge and count all edges incident to that particular edge, then we see that it is equal to $p-1+p-1-1-1=2(p-2)$.

Thus, the line graph formed with a complete graph of order $(p-1)$ is a $2(p-2)$ regular graph of order $\binom{p}{2}$. The definition of the conjugate graph of a K-metacyclic group, along with the above two claims, provides a proof for the theorem.

Remark 3.1: Theorem 3.7 can further be generalized to find the line graph of any finite group. Consider the conjugate graph Γ_G^c of any finite group, say $\Gamma_G^c = K_i \cup K_j \cup \ldots \cup K_t$, where $2 \leq i \leq j \leq \cdots \leq t$ and any $K_r (r=i,j,\ldots,t)$ is a complete graph of order r. Then, the line graph of the conjugate graph is either a null graph, a regular graph, or a union of one or more regular graphs and/or a null graph. i.e., $L(\Gamma_G^c) = X_i \cup X_j \cup \ldots \cup X_t$, $i \leq j \leq \cdots \leq t$ where any X_r is a $2(r-2)$-regular graph of order $\binom{r}{2}$.

Theorem 3.8: The complement graph of the conjugate graph of a K-metacyclic group is a complete multipartite graph $K_{p-1,p,p,\ldots,p\{(p-2)\text{ times}\}}$.

Proof: Since the conjugate graph of the K-metacyclic group is a disjoint union of $(p-1)$ complete graphs (Theorem 3.3), thus, in its complement graph, every vertex in each component is connected to every vertex in every other component except vertices in its own component. Thus, going in this manner, we find that the complement graph of the conjugate graph of a K-metacyclic group is the complete multipartite graph, $K_{p-1,p,p,\ldots,p\{(p-2)\text{ times}\}}$.

Remark 3.2: Theorem 3.8 can be further generalized to find the complement graph of the conjugate graph of any finite group.

Consider the conjugate graph Γ_G^c of any finite group, say $\Gamma_G^c = K_i \cup K_j \cup \ldots \cup K_t$, where $2 \le i \le j \le \cdots \le t$ and any $K_r (r = i, j, \ldots, t)$ is a complete graph of order r. Then, the complement graph $\overline{\Gamma_G^c}$ is always a complete multipartite graph and is given by, $\overline{\Gamma_G^c} = \mathrm{K}_{\mathrm{i,j,\ldots,t}}$.

Theorem 3.9: The Laplacian of the conjugate graph of a K-metacyclic group of order $p(p-1)$ is:

$$L_{i,j} = \begin{cases} p-2, & \text{if } i = j \text{ and } v_i \in C.C(1) \\ p-1, & \text{if } i = j \text{ and } v_i \notin C.C(1) \\ -1, & \text{if } i \neq j \text{ and } v_i, v_j \in C.C(k), \text{where } k = 1,2,\ldots,p-1 \\ 0, & \text{otherwise} \end{cases}$$

Proof: According to the definition of Laplacian (Merris, 1994),

$$L_{i,j} = \begin{cases} \deg(v_i), & \text{if } i = j \\ -1, & \text{if } i \neq j, v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

For the conjugate graph of a K-metacyclic group, $\deg(v_i) = p - 2$ if v_i belongs to C.C(1) (referring to theorem 3.1 and theorem 3.3) since the graph component formed from C.C(1) is a complete graph of order $p-1$. Similarly, the value of $\deg(v_i) = p - 1$ if it belongs to any other conjugacy class.

Now, we know that any vertex v_i, in the graph is adjacent to any other vertex v_j if and only if v_i and v_j belong to a specific conjugacy class. Combining the definition of Laplacian and the above statements, we get,

$$L_{i,j} = \begin{cases} p-2, & \text{if } i = j \text{ and } v_i \in C.C(1) \\ p-1, & \text{if } i = j \text{ and } v_i \notin C.C(1) \\ -1, & \text{if } i \neq j \text{ and } v_i, v_j \in C.C(k), \text{where } k = 1,2,\ldots,p-1 \\ 0, & \text{otherwise} \end{cases}$$

Theorem 3.10: The girth of the conjugate graph of a K-metacyclic group is 3.

Proof: We know that the conjugate graph is a union of complete graphs, and hence, we always get a cycle of length 3. Thus, its girth is 3.

Conjecture 3.1: The spectrum of the adjacency matrix of a conjugate graph of a K-metacyclic group is of the form: $\{-1^{p(p-2)}, p-2, (p-1)^{p-2}\}$

Examples: Listing the spectrum of the conjugate graph of a K-metacyclic group for small values of p with the help of a computer we get,

When $p = 3$:
Spectrum $= \{-1^{(15)}, 3, 4^{(3)}\}$
When $p = 5$:
Spectrum $= \{-1^{(15)}, 3, 4^{(3)}\}$
When $p = 7$:
Spectrum $= \{-1^{(35)}, 5, 6^{(5)}\}$
When $p = 11$:
Spectrum $= \{-1^{99}, 9, 10^{9}\}$

4. Conclusion

In our study, we have obtained the conjugate graph of a K-metacyclic group G, with defining relations, $a^p = b^{p-1} = 1, b^{-1}ab = a^r, (r-1, p) = 1$, where r is the primitive root modulo p and p is an odd prime. We denote this graph by Γ_G^c. There are $p-1$ noncentral conjugacy classes of G irrespective of the values of r. The conjugate graph of the K-metacyclic group G presented above is $K_{p-1} \cup_{p-2} K_p$. The graph is found to be a perfect graph, and the girth of the graph is always 3. The graph is planar only for $p = 3$, otherwise it is nonplanar. The line graph of Γ_G^c is a union of regular graphs having $p-1$ components of a different order. The complement graph of Γ_G^c is a complete multipartite graph.

List of symbols

G	K-metacyclic group with defining relations, $a^p = b^{p-1} = 1, b^{-1}ab = a^r, (r-1, p) = 1$
Z	Center of a group
K_n	Complete graph of order n
Γ_G^c	Conjugate graph of G
$L(\Gamma_G^c)$	Line graph of Γ_G^c
$\overline{\Gamma_G^c}$	Complement graph of Γ_G^c
$\chi(G)$	Chromatic number of G
$\omega(G)$	Clique number of G
$\alpha(G)$	Independence nymber of G
$\gamma(G)$	Dominating number of G

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Compliance with ethical standards

Conflict of interest

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