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Exact solutions of classes of second order nonlinear partial differential equations reducible to first order

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ABSTRACT

This paper illustrates the successful implementation of the method of variation of parameters in combination with the method of characteristics and other techniques to obtain exact solutions for a wide range of partial differential equations. The proposed approach reduces partial differential equations (PDEs) to first-order differential equations, referred to as classical equations, including Bernoulli, Ricatti, and Abel equations. In addition, the techniques proposed have the ability to produce precise solutions for nonlinear second order PDEs. For each PDE class, the method's effectiveness is demonstrated through illustrative examples.

1. Introduction

Partial differential equations are widely used as models to describe complex physical phenomena in various fields of science, especially in fluid mechanics, solid state physics, and plasma waves. Nonlinear differential equations also include optics, bio-hydrodynamics, and nonlinear quantum mechanics (Dubey et al., 2020; Guo et al., 2020).

In recent years, we have seen significant progress in the development of effective methods for obtaining exact solutions of nonlinear partial differential equations, such as the inverse scattering transform (Ablowitz and Clarkson, 1991), the extended Tanh function method (Fan, 2000), the truncated expansion method (Kudryashov and Loguinova, 2008), the F-expansion method (Zhang, 2005a; 2005b), the Jacobi elliptic method (Shikuo et al., 2001), the Backlund transformations (Lu et al., 2006), the sine-cosine function method (Wazwaz, 2004), the (G'/G) expansion method (Wang et al., 2008) and the extended Kudryashov method (Hassan et al., 2014).

One of the effective tools to solve PDEs is the method of characteristics (Higgins, 2019; Myint and Debnath, 2007; Zachmanoglou and Thoe, 1986; Rhee et al., 1986). The core idea of this method is to reduce the partial differential equation and give initial conditions to a set of differential equations that can be easily and directly solved resulting in general exact solutions for the initial problem.

The method of variation of parameters (Hounkonnou and Sielenou, 2009; Polyanin and Manzhirov, 2006; Kečkić Jovan, 1976; Olver, 2014; Kevorkian, 1990) can also be successfully used in some cases to reduce by one the order of nonlinear partial differential equations. By reducing the order of these equations, suitable analytical methods can be applied for resolution.

However, there are certain classes of nonlinear second-order partial differential equations that cannot be exactly solved using classical methods. Therefore, it is necessary to resort to novel methods to solve a wide variety of PDEs.

In this paper, we begin by presenting the relatively well-known results, focusing on the solutions of linear PDEs in section 2. Subsequently, in section 3, we establish the exact solutions of nonlinear PDEs. These ideas will then be extended to more complex cases and several examples will be provided for illustrative purposes.

In sections 4 and 5, we solve some classes of PDEs called Bernoulli type and Ricatti Model:

$$u_t + a(x,t)u_x = b(x,t)u + \alpha(x,t)u^n,$$

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and

$$u_t + a(x,t)u_x = b(x,t)u(x,t) + \alpha(x,t) + \beta(x,t)u^2.$$

In sections 6, 7, and 8, we demonstrate the successful implementation of the variation of parameters in combination with the characteristic method to obtain the exact solutions of the nonlinear second-order partial differential equations of the form:

$$u_{tt} + a(x,t)u_{xt} = b(x,t) + (u_t + a(x,t)u_x)f(u),$$
$$u_{xt} + a(x,t)u_{xx} = b(x,t) + (a(x,t)u_x + u_t)f(u),$$
$$f'(u_t)(u_{tt} + au_{xt}) = B(x,t) + A(u)(u_t + au_x).$$

In the final section, we extend the proposed method to general classes of nonlinear second order partial differential equations of the form:

$$u_{tt} + a(x,t)u_{xt} + b(u)u_t(u_t + a(x,t)u_x) = \alpha(x,t)e^{-\int b(u)du} + G(u)(u_t + a(x,t)u_x).$$

The special cases $b(u) = -\frac{1}{u}$ and $G(u) = \beta u^n$ have been investigated by Hounkonnou and Sielenou (2009). We apply our approach to find more general exact solutions.

2. Partial differential equations of the form $u_t + a(x,t)u_x - \alpha(x,t)u = b(x,t)$

We consider the first order partial differential equation of the following form:

$$\begin{cases} u_t + a(x,t)u_x - \alpha(x,t)u = b(x,t) \\ u(x,0) = \phi(x). \end{cases} \quad (1)$$

where, u is a function of $(x,t) \in \mathbb{R}^2$. First, we solve the differential equation

$$\begin{cases} \frac{dx(t)}{dt} = a(x(t),t) \\ x(0) = x_0. \end{cases}.$$

Then Eq. 1 can be rewritten as

$$u_t(x(t),t) + a(x(t),t)u_x - \alpha(x(t),t)u = b(x(t),t). \quad (2)$$

We multiply Eq. 2 by $e^{-\int_0^t \alpha(x(s),s)ds}$, we get

$$\frac{d}{dt}[e^{-\int_0^t \alpha(x(s),s)ds}u] = b(x(t),t)e^{-\int_0^t \alpha(x(s),s)ds}. \quad (3)$$

Therefore, the following statement holds.

Proposition 1: The first order partial differential Eq. 1 can be reduced to the differential Eq. 3 and the solution of Eq. 1 is

$$u(x,t) = \phi(x_0)e^{\int_0^t \alpha(x(s),s)ds} + e^{\int_0^t \alpha(x(s),s)ds}\int_0^t b(x(\tau),\tau)e^{-\int_0^\tau \alpha(x(s),s)ds}d\tau.$$

For illustration, let us consider this example.

Example 1: Let $a(x,t) = x$, $\alpha(x,t) = 1$ and $b(x,t) = x + t$.

$$\begin{cases} u_t + xu_x - u = x + t \\ u(x,0) = \phi(x). \end{cases} \quad (4)$$

Let $x(t)$ be the solution of $\begin{cases} \frac{dx(t)}{dt} = x \\ x(0) = x_0 \end{cases}$, then $x(t) = x_0e^t$. Eq. 4 can be rewritten as

$$\frac{d}{dt}u(x(t),t) = u(x(t),t) + x + t.$$

Then

$$\frac{du}{dt} - u = x + t,$$
$$e^{-t}\frac{du}{dt} - e^{-t}u = (x(t) + t)e^{-t},$$
$$\frac{d}{dt}(e^{-t}u) = (x(t) + t)e^{-t}.$$

Therefore

$$u(x(t),t) = e^t\phi(x_0) + e^t\int_0^t (x(s) + s)e^{-s}ds = e^t\phi(xe^{-t}) + e^t\int_0^t (x_0e^s + s)e^{-s}ds.$$

The solution of Eq. 4 is

$$u(x,t) = e^t\phi(xe^{-t}) + xt - t - 1 + e^t.$$

Fig. 1 shows the solution of Eq. 4 with $\phi(x) = x^2$.

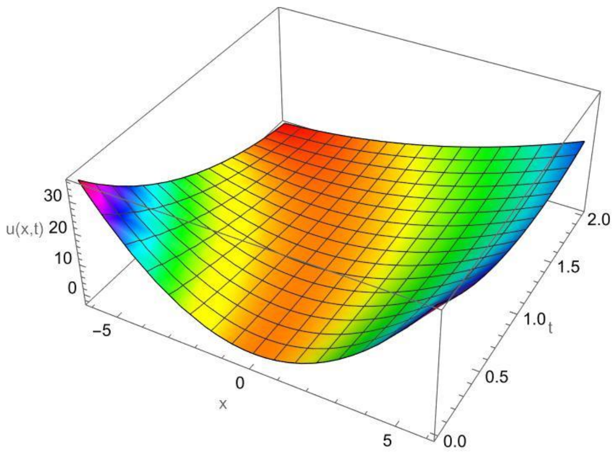

Fig. 1: Solution of Eq. 4 with $\phi(x) = x^2$

3. Partial differential equations of the form $u_t + a(x,t)u_x = f(u)b(x,t)$

Let us consider the first order partial differential equation of the following type:

$$\begin{cases} u_t + a(x,t)u_x = f(u)b(x,t) \\ u(x,0) = \phi(x). \end{cases} \quad (5)$$

Let $x(t)$ be the solution of $\begin{cases} \frac{dx(t)}{dt} = a(x(t),t) \\ x(0) = x_0. \end{cases}$. Then Eq. 5 is transformed to

$$\frac{d}{dt}u(x(t),t) = f(u(x(t),t))b(x(t),t),$$

and

$$\frac{du}{f(u)} = b(x(t), t)dt. \quad (6)$$

Then we deduce the following result.

Proposition 2: The first order partial differential Eq. 5 can be reduced to Eq. 6. Furthermore, let $F(u) = B(t)$ where $B(t) = b(x(t), t)$ and $F(u(x_0, 0)) = b(x_0, 0)$, then the general solution of Eq. 5 is easily determined by solving the equation

$$F(u) = F(\phi(x_0)) + \int_0^t b(x(s), s)ds.$$

As an illustration, let us consider the following examples.

Example 2: Let $a(x,t) = x$, $f(u) = u^2$ and $b(x,t) = 1$.

$$\begin{cases} u_t + xu_x = u^2 \\ u(x,0) = x. \end{cases} \quad (7)$$

Since $\frac{du}{u^2} = 1$, we get

$$u(x(t), t) = \frac{1}{-t + \frac{1}{xe^{-t}}}.$$

Fig. 2 shows the solution of Eq. 7 with $\phi(x) = x$.

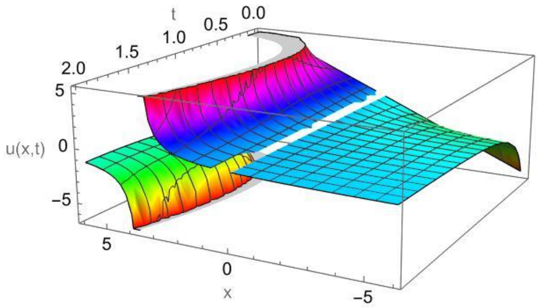

Fig. 2: Solution of Eq. 7 with $\phi(x) = x$

Example 3: Let $a(x,t) = x$, $f(u) = u^2 + 1$ and $b(x,t) = x + t^2$.

$$\begin{cases} u_t + xu_x = (u^2 + 1)(x + t^2) \\ u(x,0) = \phi(x). \end{cases} \quad (8)$$

We get

$$\frac{du}{dt} = (u^2 + 1)(x + t^2),$$

$$\tan^{-1}(u) = \tan^{-1}(\phi(xe^{-t})) + \int_0^t (x_0 e^s + s^2)ds.$$

Then

$$u(x,t) = \tan(\tan^{-1}(\phi(xe^{-t})) + x(1 - e^{-t}) + \frac{t^3}{3}).$$

Fig. 3 shows the solution of Eq. 8 with $\phi(x) = x$.

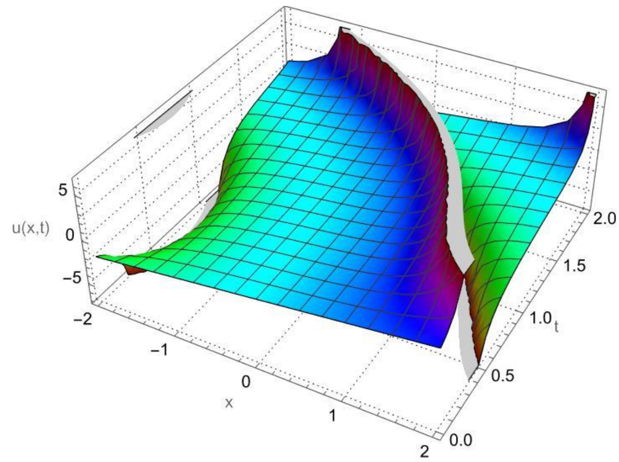

Fig. 3: Solution of Eq. 8 with $\phi(x) = x$

4. Bernoulli equation of the form $u_t + a(x,t)u_x = b(x,t)u + \alpha(x,t)u^n$

We consider the Bernoulli Equation of the form:

$$\begin{cases} u_t + a(x,t)u_x = b(x,t)u + \alpha(x,t)u^n \\ u(x,0) = \phi(x). \end{cases} \quad (9)$$

Let $x(t)$ be the solution of $\begin{cases} \frac{dx(t)}{dt} = a(x(t), t) \\ x(0) = x_0. \end{cases}$.

Let $u = u(x(t), t)$, then the general solution of Eq. 9 can be obtained by using

$$\frac{du}{dt} = b(x(t), t)u + \alpha(x(t), t)u^n,$$

and taking $v = u^{1-n}$.

Let us consider the following examples.

Example 4: *Let* $a(x,t) = x$, $b(x,t) = t$, $\alpha(x,t) = x + t$ *and* $n = 2$.

$$\begin{cases} u_t + xu_x = tu + (x + t)u^2 \\ u(x,0) = \phi(x). \end{cases} \quad (10)$$

We have $x(t) = x_0 e^t$, and

$$\frac{du}{dt} = tu + (x(t) + t)u^2,$$

where, $u = u(x(t), t)$. Let $v = u^{-1}$, we get

$$\frac{dv}{dt} + tv = -(x(t) + t),$$

and

$$\frac{d}{dt}(e^{\frac{t^2}{2}} v) = -(x(t) + t)e^{\frac{t^2}{2}},$$

$$v = -e^{-\frac{t^2}{2}} x_0 \int_0^t e^{s + \frac{s^2}{2}} ds - e^{-\frac{t^2}{2}}(e^{\frac{t^2}{2}} - 1) + \frac{e^{-\frac{t^2}{2}}}{\phi(x_0)}.$$

Then

$$u(x,t) = (-e^{-\frac{t^2}{2} - t} x \int_0^t e^{s + \frac{s^2}{2}} ds + (e^{-\frac{t^2}{2}} - 1) + \frac{e^{-\frac{t^2}{2}}}{\phi(xe^{-t})})^{-1}.$$

Fig. 4 shows the solution of Eq. 10 with $\phi(x) = x$.

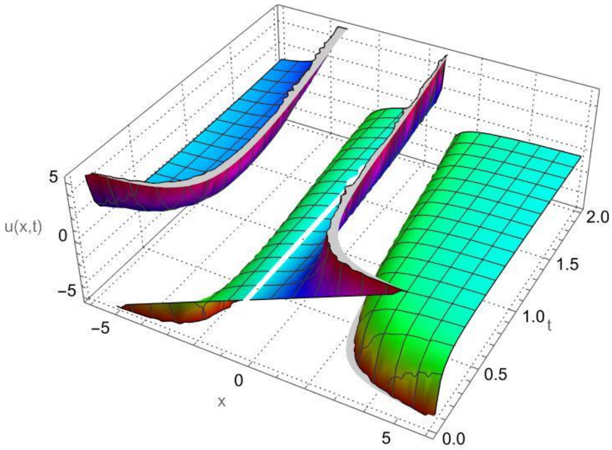

Fig. 4: Solution of Eq. 10 with $\phi(x) = x$

Example 5: Let $a(x,t) = x,\ b(x,t) = 1,\ \alpha(x,t) = t$ and $n = 2$.

$$\begin{cases} u_t + xu_x = u + tu^2 \\ u(x,0) = \phi(x). \end{cases} \tag{11}$$

Let $u = u(x(t),t)$; we get

$$\frac{du}{dt} - u = tu^2.$$

Let $v = u^{-1}$, then

$$\frac{dv}{dt} + v = -t,$$

and

$$\frac{d}{dt}(e^t v) = -te^t.$$

Hence, we obtain

$$e^t v = \int_0^t -se^s ds + v(x_0, 0),$$
$$v(x,t) = -t + 1 - e^{-t} + \frac{e^{-t}}{\phi(xe^{-t})}$$

and

$$u(x,t) = (-t + 1 - e^{-t} + \frac{e^{-t}}{\phi(xe^{-t})})^{-1}.$$

Fig. 5 shows the solution of Eq. 11 with $\phi(x) = x^2$.

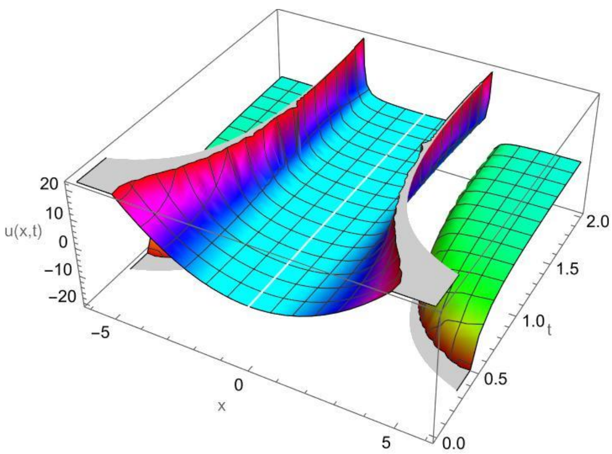

Fig. 5: Solution of Eq. 11 with $\phi(x) = x^2$

Example 6: *Let* $a(x,t) = 1,\ b(x,t) = t,\ \alpha(x,t) = x$ and $n = 2$.

$$\begin{cases} u_t + u_x = tu + xu^2 \\ u(x,0) = \phi(x). \end{cases} \tag{12}$$

Let $x(t) = t + x_0$ and $u = u(x(t),t)$,

$$\frac{du}{dt} = tu + (t + x_0)u^2.$$

Let $v = u^{-1}$, we get

$$\frac{dv}{dt} + tv = -(t + x_0),$$
$$\frac{d}{dt}(e^{\frac{t^2}{2}} v) = -(t + x_0)e^{\frac{t^2}{2}},$$

and

$$e^{\frac{t^2}{2}} v = -\int_0^t (s + x_0)e^{\frac{s^2}{2}} ds + \frac{1}{\phi(x-t)}.$$

Then

$$u(x,t) = (-1 + e^{-\frac{t^2}{2}} - (x-t)e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} ds + \frac{e^{-\frac{t^2}{2}}}{\phi(x-t)})^{-1}.$$

Fig. 6 shows the solution of Eq. 12 with $\phi(x) = x^2$.

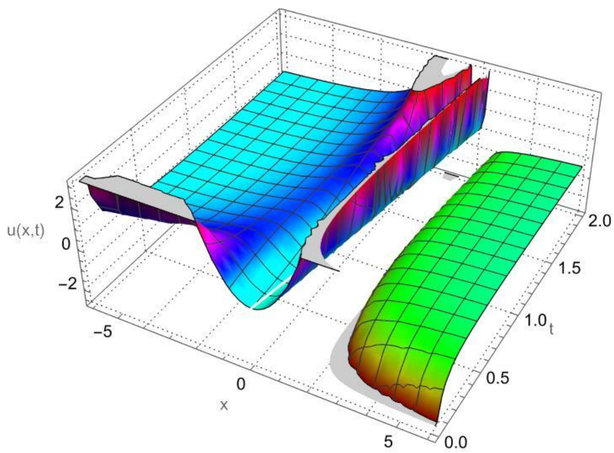

Fig. 6: Solution of Eq. 12 with $\phi(x) = x^2$

5. Riccati model $u_t + a(x,t)u_x = b(x,t)u + \alpha(x,t) + \beta(x,t)u^2$

Let u_1 be the exact solution of the Riccati Model

$$u_t + a(x,t)u_x = b(x,t)u + \alpha(x,t) + \beta(x,t)u^2,$$

then by using the substitution $u = u_1 + w$, we get the Bernoulli model

$$w_t + a(x,t)w_x = \gamma(x,t)w + \beta(x,t)w^2$$

which is solved in the previous section.

6. Partial differential equations of the form $u_{tt} + a(x,t)u_{xt} = b(x,t) + (u_t + a(x,t)u_x)f(u)$

We consider the second order partial differential equation of the following type:

$$u_{tt} + a(x,t)u_{xt} = b(x,t) + (u_t + a(x,t)u_x)f(u). \quad (13)$$

First, let $x(t)$ be the solution of

$$\begin{cases} \frac{dx(t)}{dt} = a(x(t),t) \\ x(0) = x_0. \end{cases}$$

Then Eq. 13 takes the form

$$\frac{d}{dt}\left(u_t(x(t),t)\right) = b(x,t) + (u_t + a(x,t)u_x)f(u). \quad (14)$$

We suppose that $u_t(x(t),t)) = H(t) + K(u)$. Then Eq. 14 is rewritten as

$$\frac{d}{dt}(u_t(x(t),t)) = H'(t) + K'(u)(u_t + au_x).$$

It is clear that Eq. 14 will take place if

$$\begin{cases} H'(t) = b(x(t),t) \\ K'(u) = f(u). \end{cases}$$

Hence, we have the following result.

Proposition 3: The second order partial differential Eq. 13 can be transformed to

$$\frac{d}{dt}(u_t(x(t),t)) = H'(t) + K'(u)(u_t + au_x),$$

where, the functions H and K are the general solutions of $\begin{cases} H'(t) = b(x,t) \\ K'(u) = f(u) \end{cases}$. Let us consider the following example.

Example 7: *Let $a(x,t) = x$, $f(u) = u^2$ and $b(x,t) = x + t$.*

$$u_{tt} + xu_{xt} = x + t + (u_t + xu_x)u^2.$$

First, we solve $\begin{cases} \frac{dx(t)}{dt} = x \\ x(0) = x_0 \end{cases}$. The solution is $x = x_0e^t$. The functions H and K are general solutions of

$$H'(t) = x(t) + t,$$
$$K'(u) = u^2.$$

Then we get

$$H(t) = x_0e^t + \frac{t^2}{2} + C_1,$$
$$H(t) = x + \frac{t^2}{2} + C_1,$$

and

$$K(u) = \frac{1}{3}u^3 + C_2.$$

which reduces the partial differential equation to the ODE of order one:

$$u_t(x,t) = x + \frac{t^2}{2} + \frac{u^3}{3} + C,$$

known as the Abel equation which can be solved by various methods. For more details, see Zwillinger (1998), Murphy (1960), and Kamke (1977). In Panayotounakos and Zarmpoutis (2011), the authors give implicit solutions using the first kind of Bessel's functions and the second kind of Newmann functions for the canonical form of Abel's equation.

7. Partial differential equations of the form $u_{xt} + a(x,t)u_{xx} = b(x,t) + (a(x,t)u_x + u_t)f(u)$

Let us consider the second order partial differential equation of the following type:

$$u_{xt} + a(x,t)u_{xx} = b(x,t) + (a(x,t)u_x + u_t)f(u). \quad (15)$$

We take $u_x(x(t),t) = H(t) + K(u)$, where the functions H and K are the general solutions of

$$H'(t) = b(x(t),t)$$
$$K'(u) = f(u).$$

Example 8: Consider the second order partial differential equation

$$u_{xt} + xu_{xx} = x + t + (u_t + xu_x)u^2.$$

In Eq. 15, we take $b(x,t) = x + t$, $a(x,t) = x$ and $f(u) = u^2$.

The functions H and K are the general solutions of

$$H'(t) = x_0t + t$$
$$K'(u) = u^2.$$

Then

$$u_x(x,t) = x + \frac{t^2}{2} + \frac{u^3}{3} + C.$$

The latter is Abel's equation which is integrable by various methods known in the literature. See Zwillinger (1998), Murphy (1960), and Kamke (1977) for more details

8. Partial differential equations of the form $f'(u_t)(u_{tt} + au_{xt}) = B(x,t) + A(u)(u_t + au_x)$

Consider the non linear second order differential equation of the general form

$$f'(u_t)(u_{tt} + au_{xt}) = B(x,t) + A(u)(u_t + au_x). \quad (16)$$

Let $f(u_t)$ of the form $H(t) + K(u)$, then

$$\frac{d}{dt}(f(u_t)) = H'(t) + K'(u)(u_t + au_x)$$
$$f'(u_t)(u_{tt} + au_{xt}) = H'(t) + K'(u)(u_t + au_x). \quad (17)$$

Therefore, the following statement holds.

Proposition 4: The general solution of the non linear second order differential Eq. 16 is obtained by solving Eq. 17 where the functions H and K are the general solutions of $H'(t) = B(x(t),t)$ *and* $K'(u) = A(u)$.

9. Partial differential equations of the form $u_{tt} + au_{xt} + b(u)u_t(u_t + au_x) = \alpha(x,t)e^{-\int b(u)du} + G(u)(u_t + au_x)$

Consider the non linear second order partial differential equation of the general form

$$u_{tt} + au_{xt} + b(u)u_t(u_t + au_x) = \alpha(x,t)e^{-\int b(u)du} + G(u)(u_t + au_x). \quad (18)$$

In Hounkonnou and Sielenou (2009), the author investigated special cases of Eq. 18 when u and α are functions of one variable. Multiplying both sides of Eq. 18 by $e^{\int b(u)du}$, we get

$$(u_{tt} + au_{xt})e^{\int b(u)du} + b(u)u_t(u_t + au_x)e^{\int b(u)du} = \alpha(x,t) + G(u)(u_t + au_x)e^{\int b(u)du}$$

then

$$\frac{d}{dt}[u_t(x(t),t)e^{\int b(u)du}] = \alpha(x,t) + G(u)(u_t + au_x)e^{\int b(u)du}$$

hence, the nonlinear second order differential Eq. 18 is easily solved if we suppose that

$$u_t = (H(t) + K(u))e^{-\int b(u)du} \quad (19)$$

then,

$$H'(t) + K'(t)(u_t + au_x) = \alpha(x,t) + G(u)(u_t + au_x)e^{\int b(u)du}$$

where, the functions H and K are solutions of

$$H'(t) = \alpha(x(t),t)$$
$$K'(u) = G(u)e^{\int b(u)du}.$$

We obtain the following result.

Proposition 5: The solution of the nonlinear second order differential Eq. 18 is obtained by solving Eq. 19 where the functions H and K are the general solutions of $H'(t) = \alpha(x(t),t)$ *and* $K'(u) = G(u)e^{\int b(u)du}$

Example 9: Let us consider the nonlinear second order differential Eq. 18 with

$b(u) = -\frac{1}{u}$, $G(u) = 2u^2$, $\alpha = x + t$ and $a = x$,

then

$$\begin{cases} u_{tt} + xu_{xt} - \frac{u_t}{u}(u_t + xu_x) = (x+t)u + 2u^2(u_t + xu_x) \\ u(x,0) = x \\ u_t(x,0) = x^2 + x^3 \end{cases}. \quad (20)$$

First, let $x(t)$ be the solution of

$$\begin{cases} \frac{dx(t)}{dt} = x \\ x(0) = x_0. \end{cases}$$

Then $x(t) = x_0 e^t$. Taking

$$H'(t) = \alpha(x(t),t) = x_0 e^t + t,$$

and

$$K'(u) = 2u^2(\tfrac{1}{u}) = 2u$$

we get

$$H(t) = x_0 e^t + \frac{t^2}{2}$$

and

$$K(u) = u^2.$$

We obtain

$$u_t = (x + \frac{t^2}{2} + u^2 + c)u,$$

which can be reduced to the usual canonical form of Abel's equation of the first kind

$$w'_t(x,t) = w^3(x,t) + k(x,t). \quad (21)$$

The latter is integrable by various methods known in the literature. See Zwillinger (1998), Murphy (1960), and Kamke (1977) for a good compilation of techniques developed to solve Eq. 21 for particular expressions of $k(x,t)$.

If we take the initial condition $u(x,0) = x$ and

$$u_t(x,0) = x^2 + x^3,$$

then

$$u_t(x,t) = (x + \frac{t^2}{2} + u^2)u.$$

Fig. 7 shows the solution of Eq. 20 with $u(x,0) = x$ and $u_t(x,0) = x^2 + x^3$.

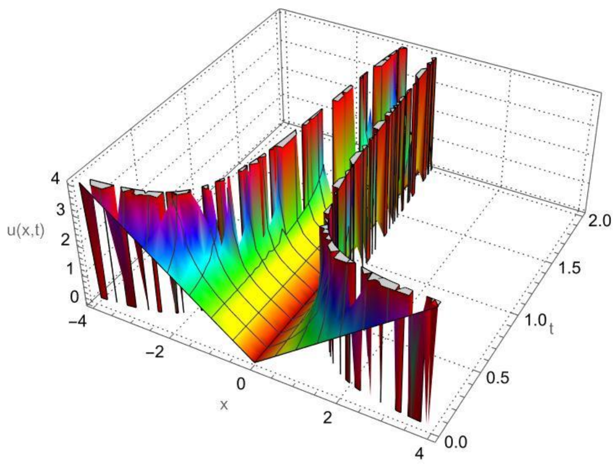

Fig. 7: Solution of Eq. 20 with $u(x,0) = x$ and $u_t(x,0) = x^2 + x^3$

In Eq. 18, we take $b(u) = -\frac{1}{u}$ then

$u_{tt} + a(x,t)u_{xt} - \frac{u_t}{u}(u_t + a(x,t)u_x) = \alpha(x,t)u + G(u)(u_t + a(x,t)u_x).$

If $G = u^n$,

$$u_{tt} + a(x,t)u_{xt} - \frac{u_t}{u}(u_t + a(x,t)u_x) = \alpha(x,t)u + u^n(u_t + a(x,t)u_x). \quad (22)$$

Therefore, we state the following result.

Proposition 6: The exact solution of the non linear second order partial differential Eq. 22 is

$$u = [-e^{-n\int H(t)dt} \int e^{n\int H(t)dt} dt]^{-\frac{1}{n}}.$$

where, the function H is the general solution of $H'(t) = \alpha(x(t),t)$.

Proof: Eq. 19 can be rewritten as

$$u_t = \big(H(t) + K(u)\big)u. \quad (23)$$

where, $H'(t) = \alpha(x(t),t)$ and $K'(u) = u^{n-1}$. Substituting $K(u) = \frac{u^n}{n}$ to Eq. 23, we get

$$u_t = H(t)u + \frac{u^{n+1}}{n}$$

which is Bernoulli differential Equation where $H(t)$ is the general solution of $H'(t) = \alpha(x(t),t)$. Then we get

$$u_t - H(t)u = \frac{u^{n+1}}{n}. \quad (24)$$

Let $v = u^{-n}$ then $u = v^{-\frac{1}{n}}$ and

$$u_t = -\frac{1}{n}v_t v^{-\frac{1}{n}-1}.$$

Eq. 24 takes the form

$$-\frac{1}{n}v_t v^{-\frac{1}{n}-1} - H(t)v^{-\frac{1}{n}} = \frac{1}{n}v^{-\frac{n+1}{n}},$$

which leads to the simpler form

$$v_t + nH(t)v = -1,$$

we obtain the solution

$$v = -e^{-n\int H(t)dt} \int e^{\int nH(t)dt} dt.$$

Finally, the exact solution of the nonlinear second order partial differential Eq. 22 is determined by

$$u = [-e^{-n\int H(t)dt} \int e^{n\int H(t)dt} dt]^{-\frac{1}{n}}.$$

10. Conclusion

We investigated a wide range of partial differential equations reduced to first order by the variation of parameters method and other techniques, such as the method of characteristics, in this paper. Usually, these first-order differential equations can be transformed into well-known solvable classical differential equations. It has been demonstrated that the techniques developed in this research may be extended to various classes of nonlinear second order partial differential equations.

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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