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Moment bounds for a class of stochastic nonlinear fractional Volterra integral equations of the second kind

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ABSTRACT

This paper studies and compares the second moment (Energy growth) bounds for solutions to a class of stochastic fractional Volterra integral equations of the second kind, under some Lipschitz continuity conditions on the parameters. The result shows that both solutions exhibit exponential growth but at different rates. The existence and uniqueness of the mild solutions are established via the Banach fixed point theorem.

1. Introduction

Like fractional derivatives, fractional integral equations have recently attracted the interest of many researchers and scientists. Both linear and nonlinear Volterra integral equations of the second kind have become necessary and essential in modeling real-life (world) problems and physical phenomena in applied Mathematics, Physics, Sciences, and Engineering (Berenguer et al., (2010). Particularly, the fractional integral equations have often found their applications in heat transformations and heat radiation, population growth models, biological species living together, porous media, rheology, control, electrochemistry, viscoelasticity, electromagnetism fluid structure, coupling, and particle mechanics (Agarwal et al., 2015; Hamdan et al., 2019). In addition, they have been applied in stochastic fractional differential and integral equations (Omaba, 2021a; 2021b; Omaba and Enyi, 2021).

Motivated by the above numerous modeling applications of fractional Volterra integral equations, we consider the following stochastic fractional nonlinear Volterra integral equations of the second kind:

$$\varphi(t) = \theta(t) + \frac{\delta}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} k(t,s)\vartheta(s,\varphi(s))\dot{w}(s)ds$$
$$= \theta(t) + \frac{\delta}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} k(t,s)\vartheta(s,\varphi(s))dw(s), \qquad (1)$$
$$0<\alpha<1,\ 0<a\le t\le b<\infty,$$

where, $k(t,s)$ is assumed to be a convolution kernel (also called a displacement kernel) given by $k(t,s) = k(t-s) = e^{-(t-s)}$, $\vartheta: [a,b]\times\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous on the second variable, $\theta: [a,b]\to\mathbb{R}$ is continuous, $\dot{w}$ is a Gaussian white noise process and δ is a positive parameter called the level of the noise term; and,

$$\varphi(t) = \theta(t) + \frac{\delta}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} k(t,s,\varphi(s))\dot{w}(s)ds$$
$$= \theta(t) + \frac{\delta}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} k(t,s,\varphi(s))dw(s), \qquad (2)$$

with $0<\alpha<1$, $0<a\le t\le b<\infty$, and $k: [a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ assumed to be Lipschitz continuous on the third variable. Here, φ is the unknown function.

Remark 1.1: Though the integral Eqs. in 1, and 2 are defined for all $0<\alpha<1$, their existence and uniqueness and growth moment bound results to hold only for $\alpha\in\left(\frac{1}{2},1\right)$. This is because, in the proofs of the main results we have the gamma function $\Gamma(2\alpha-1)$, which is defined only for $2\alpha-1>0$ and hence $\alpha\in\left(\frac{1}{2},1\right)$; and also, Proposition 2.10 and Proposition 2.13 apply only for $\rho=2\alpha-1>0$. Thus, for $\alpha\in\left(0,\frac{1}{2}\right)$, the solution(s) will fail to exist, and consequently, no moment bounds.

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Definition 1.2: The unknown function $\{\varphi(t),\ a \le t \le b\}$ is called a mild solution if almost surely, φ satisfies Eq. 1, and Eq. 2 respectively.

If in addition, $\{\varphi(t),\ a \le t \le b\}$ satisfies the following $\sup_{t\in[a,b]} \boldsymbol{E}|\varphi(t)|^2 < \infty$, then we say that $\{\varphi(t),\ a \le t \le b\}$ is a random field solution to Eq. 1, and Eq. 2 respectively.

The organization of the paper is as follows. Section 2 contains the preliminaries and in Section 3, we gave the main results: Proofs of results for Eq. 1 in Subsection 3.1 and proofs of results for Eq. 2 in Subsection 3.2. Section 4 contains a short summary of the paper.

2. Preliminaries

In general, the Volterra integral equation can be written,

$$v(t)u(t) = \omega(t) + \mu \int_a^t k(t,s)u(s)ds.$$

It is the first kind when $v(t) = 0$ and the second kind when $v(t) = 1$ (Wazwaz, 2011). In this paper, we will be considering the Volterra integral equation of the second kind. The general form of the second kind linear Volterra integral equation is given by,

$$u(t) = \omega(t) + \mu \int_a^t k(t,s)u(s)ds, \quad a \le t \le b, \tag{3}$$

where $k(t,s)$ is called the kernel or the nucleus or the free term of the integral equation, μ is a constant parameter and $u(t)$ is the unknown function to be determined.

Theorem 2.1: (Wazwaz, 2011) If the function $k(t,s)$ is continuous in $a \le t,s \le b$ and the function $\omega(t)$ is continuous in $a \le t \le b$, then there is a unique continuous solution of the integral Eq. 3.

Now, we define a generalized derivative for a deterministic function w:

Definition 2.2: Suppose $f(t)$ is any smooth and compactly supported function. Then the generalized derivative $\dot{w}(t)$ of $w(t)$ (not necessarily differentiable) is given by,

$$\int_0^\infty f(t)\dot{w}(t)dt = -\int_0^\infty \dot{f}(t)w(t)dt.$$

Therefore,

$$\int_0^t f(s)\dot{w}(s)ds = f(t)w(t) - \int_0^t \dot{f}(s)w(s)ds.$$

Next, we present the following estimates (bounds) on an incomplete gamma function:

Theorem 2.3: (Neuman, 2013) Let $x > 0$, then the following inequalities:

$$exp\left(\frac{-a\,x}{a+1}\right) \le \frac{a}{x^a}\gamma(a,x) \le {}_1F_1(a; a+1; -x) \le \frac{1}{a+1}(1 + ae^{-x}),$$

hold, where ${}_1F_1(a; a+1; -x)$ is a confluent hypergeometric (Kummer) function.

More so, for $0 < a \le 1$,

$$\frac{1-e^{-x}}{x} \le \frac{a}{x^a}\gamma(a,x).$$

Lemma 2.4: (Natalini and Palumbo, 2000) For $a > 1$, $B > 1$ and $x > \frac{B}{B-1}(a-1)$ we have,

$$x^{a-1}e^{-x} < |\Gamma(a,x)| < Bx^{a-1}e^{-x}.$$

Remark 2.5: From the above results, $\gamma(z,x)$ is the incomplete gamma function and $\Gamma(z,x)$ is the complement of the incomplete gamma function satisfying the relation,

$$\gamma(z,x) = \Gamma(z) - \Gamma(z,x),$$

with $\Gamma(z)$ the Euler's gamma function.

3. Main results

Here, we make global Lipschitz continuity conditions on $\vartheta(.,\varphi)$ and $k(,.\varphi)$ as follows:

Condition 3.1: Let $0 < Lip_\vartheta < \infty$. Then for all $x, y \in \mathbb{R}$ and $t \in [a,b]$,

$$|\vartheta(t,x) - \vartheta(t,y)| \le Lip_\vartheta|x-y|.$$

We set $\vartheta(t,0) = 0$ for convenience only.

Condition 3.2: Suppose $0 < M < \infty$. Then for all $x, y \in \mathbb{R}$ and $t, s \in [a,b]$,

$$|k(t,s,x) - k(t,s,y)| \le M|x-y|.$$

Also, set $\vartheta(t,0) = 0$ for the purpose of convenience.

Next, we define the $L^2(P)$ norm of φ by $||\varphi||_2^2 = \sup_{t\in[a,b]} E|\varphi(t)|^2$.

3.1. Proofs of results for Eq. 1

Theorem 3.3: Suppose $\alpha > \frac{1}{2}$ and Condition 3.1 holds. Then for some positive constants $c_2, \delta, Lip_\vartheta$ such that $c_2 < \frac{1}{(\delta\ Lip_\vartheta)^2}$, there exists a unique solution to Eq. 1.

The proof of the above is via the Banach fixed point theorem. First, define the operator:

$$\Omega\varphi(t) = \theta(t) + \frac{\delta}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\, e^{-(t-s)}\vartheta(s,\varphi(s))dw(s).$$

Then the solution of Eq. 1 will be obtained as a fixed point of the operator Ω.

Lemma 3.4: Let φ be a mild solution of Eq. 1 satisfying $||\varphi||_2 < \infty$. Suppose Condition 3.1 holds, then for $\alpha > \frac{1}{2}$,

$$||\Omega\varphi||_2^2 \leq c_1 + c_2\delta^2 Lip_\vartheta^2 ||\varphi||_2^2,$$

where, $c_2 := \frac{(b-a)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}$.

Proof. Applying Itô isometry and Condition 3.1, we obtain:

$$\boldsymbol{E}|\Omega\varphi(t)|^2 \leq |\theta(t)|^2 + \frac{\delta^2}{\Gamma^2(\alpha)}\int_a^t (t-s)^{2\alpha-2}\, e^{-2(t-s)}\boldsymbol{E}|\vartheta(s,\varphi(s))|^2 ds$$

$$\leq |\theta(t)|^2 + \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)}\int_a^t (t-s)^{2\alpha-2}\, e^{-2(t-s)}\boldsymbol{E}|\varphi(s)|^2 ds$$

$$\leq |\theta(t)|^2 + \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)}||\varphi||_2^2 \int_a^t (t-s)^{2\alpha-2}\, e^{-2(t-s)} ds$$

$$\leq |\theta(t)|^2 + \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)}||\varphi||_2^2 \frac{1}{2^{2\alpha-1}}[\Gamma(2\alpha-1) - \Gamma(2\alpha-1, 2(t-a))]$$

$$\leq |\theta(t)|^2 + \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)}||\varphi||_2^2 \frac{1}{2^{2\alpha-1}}\gamma(2\alpha-1, 2(t-a)),\ \Re(\alpha) > \frac{1}{2}.$$

Using the estimate of Theorem 2.3 for $\alpha > \frac{1}{2}$ and $t > a$, one gets:

$$E|\Omega\varphi(t)|^2 \leq |\theta(t)|^2 + \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)}||\varphi||_2^2 \frac{1}{2^{2\alpha-1}}\frac{2^{2\alpha-1}(t-a)^{2\alpha-1}}{2\alpha(2\alpha-1)}(1 + (2\alpha-1)e^{-2(t-a)})$$

$$= |\theta(t)|^2 + \frac{\delta^2 Lip_\vartheta^2}{2\alpha(2\alpha-1)\Gamma^2(\alpha)}||\varphi||_2^2 (t-a)^{2\alpha-1}(1 + (2\alpha-1)e^{-2(t-a)}).$$

Now, take supremum over $t \in [a,b]$ of both sides to obtain:

$$||\Omega\varphi||_2^2 \leq c_1 + \frac{\delta^2 Lip_\vartheta^2}{2\alpha(2\alpha-1)\Gamma^2(\alpha)}||\varphi||_2^2 (t-a)^{2\alpha-1}(1 + (2\alpha-1),$$

and the result readily follows. The last inequality follows because $e^{-2(t-a)} \leq 1$ since $t-a \geq 0$.

Lemma 3.5: Suppose φ and ϕ are mild solutions of Eq. 1 satisfying $||\varphi||_2 + ||\phi||_2 < \infty$. Given that Condition 3.1 holds, then for $\alpha > \frac{1}{2}$,

$$||\Omega\varphi - \Omega\phi||_2^2 \leq c_2\delta^2 Lip_\vartheta^2 ||\varphi - \phi||_2^2.$$

Proof. The proof follows the steps of proof of Theorem 3.4.

Proof of Theorem 3.3: Let $\varphi(t) = \Omega\varphi(t)$. Then by Lemma 3.4, we have:

$$||\boldsymbol{\varphi}||_2^2 = ||\Omega\boldsymbol{\varphi}||_2^2 \leq c_1 + c_2\delta^2 Lip_\vartheta^2 ||\varphi||_2^2$$

This gives $||\varphi||_2^2[1 - c_2\delta^2 Lip_\vartheta^2] \leq c_1$ and $||\varphi||_2 < \infty$ for all $c_2 < \frac{1}{(Lip_\vartheta)^2}$.

Similarly, from Lemma 3.5, one has:

$$||\varphi - \phi||_2^2 = ||\Omega\varphi - \Omega\phi||_2^2 \leq c_2\delta^2 Lip_\vartheta^2 ||\varphi - \phi||_2^2,$$

and $||\varphi - \phi||_2^2[1 - c_2\delta^2 Lip_\vartheta^2] \leq 0$. Thus, for $c_2 < \frac{1}{(Lip_\vartheta)^2}$, we have $||\varphi - \phi||_2 \leq 0$ and the uniqueness result follows by the Banach contraction principle.

Remark 3.6: Next, we extend the above results to pth moment for all $p \geq 2$. For $p \geq 2$, define $||\varphi||_p^p = \sup_{t\in[a,b]} E|\varphi(t)|^p$.

We follow the same line of argument of proofs of Lemma 3.3 and Lemma 3.4 in Foondun and Khoshnevisan (2009).

Lemma 3.7: For $p \geq 2$, let φ be a mild solution of Eq. 1 such that $||\varphi||_p < \infty$. Suppose Condition 3.1 holds, then for $\alpha > \frac{1}{2}$,

$$||\Omega\varphi||_p^p \leq c_p + c_{\alpha,p}||\varphi||_p^p,$$

where, $c_p := 2^{p-1}c$ and $c_{\alpha,p} := 2^{p-1}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^p \frac{(b-a)^{p(\alpha-\frac{1}{2})}}{(2\alpha-1)^{\frac{p}{2}}}$ with z_p the optimal constant in the Burkholder-Davis-Gundy (BDG) inequality.

Proof. Applying BDG inequality, we have:

$$E|\Omega\varphi(t)|^p \leq 2^{p-1}|\theta(t)|^p + 2^{p-1}\left(\frac{\delta z_p}{\Gamma(\alpha)}\right)^p E\left|\int_a^t (t-s)^{2\alpha-2}\, e^{-2(t-s)}|\vartheta(s,\varphi(s))|^2 ds\right|^{\frac{p}{2}}$$

$$\leq 2^{p-1}|\theta(t)|^p + 2^{p-1}\left(\frac{\delta z_p}{\Gamma(\alpha)}\right)^p \left[\int_a^t (t-s)^{2\alpha-2}\, e^{-2(t-s)}\boldsymbol{E}|\vartheta(s,\varphi(s))|^2 ds\right]^{\frac{p}{2}}$$

$$\leq 2^{p-1}|\theta(t)|^p + 2^{p-1}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^p \left[\int_a^t (t-s)^{2\alpha-2}\, e^{-2(t-s)}\boldsymbol{E}|\varphi(s)|^2 ds\right]^{\frac{p}{2}}$$

$$\leq 2^{p-1}|\theta(t)|^p + 2^{p-1}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^p \left(\sup_{a\leq s\leq t}\boldsymbol{E}|\varphi(s)|^2\right)^{\frac{p}{2}} \left[\int_a^t (t-s)^{2\alpha-2}\, e^{-2(t-s)} ds\right]^{\frac{p}{2}}$$

$$\leq 2^{p-1}|\theta(t)|^p + 2^{p-1}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^p ||\varphi||_p^p \left[\frac{1}{2^{2\alpha-1}}\gamma(2\alpha-1, 2(t-a))\right]^{\frac{p}{2}},\ \Re(\alpha) > \frac{1}{2}.$$

Let $\sup_{t\in[a,b]}|\theta(t)|^p \leq c$. Now, take supremum over $t \in [a,b]$ and apply the estimate of Theorem 2.3 for $\alpha > \frac{1}{2}$ and $t > a$, to get:

$$||\Omega\varphi||_p^p \leq 2^{p-1}\sup_{t\in[a,b]}|\theta(t)|^p + 2^{p-1}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^p \frac{(b-a)^{p(\alpha-\frac{1}{2})}}{(2\alpha-1)^{\frac{p}{2}}}||\varphi||_p^p,$$

and the result follows.

Lemma 3.8: For $p \geq 2$, let φ and ϕ be mild solutions of Eq. 1 satisfying $||\varphi||_p + ||\phi||_p < \infty$. Suppose Condition 3.1 holds, then for $\alpha > \frac{1}{2}$,

$$||\Omega\varphi - \Omega\phi||_p^p \leq c_{\alpha,p}||\varphi - \phi||_p^p.$$

Using the above Lemma 3.4 and Lemma 3.5, we obtain the existence and uniqueness theorem.

Theorem 3.9: Let $\alpha > \frac{1}{2}$ and suppose Condition 3.1 holds. Then for a positive constant $c_{\alpha,p}$ such that $c_{\alpha,p} < 1$, there exists a unique solution to Eq. 1.

3.1.1. Upper moment bound

For the growth moment result, we present the following renewable inequality:

Proposition 3.10: (Foondun and Khoshnevisan, 2009) Let $\rho > 0$ and suppose that $f(t)$ is a locally integrable function satisfying.

$$f(t) \leq c_1 + \epsilon \int_0^t (t-s)^{\rho-1} f(s)ds, \ \forall \ t > 0,$$

where, $c_1 > 0$. Then we have:

$$f(t) \leq c_2 exp\left(c_3(\Gamma(\rho))^{\frac{1}{\rho}} \epsilon^{\frac{1}{\rho}} t\right), \text{for all } t > 0$$

for some positive numbers c_2 and c_3.

Here, is the upper moment growth bound result Assume that the function $\theta(t)$ is bounded above.

Theorem 3.11: Suppose Condition 3.1 holds. Then for all $t \in [a,b]$ we obtain:

$$E|\varphi(t)|^2 \leq c_4 exp\left(c_6 \delta^{\frac{2}{2\alpha-1}}(t-a) - 2t\right),$$

for some positive constants c_4, c_5 and,

$$c_6 = c_5 \left(\frac{\Gamma(2\alpha-1)Lip_\vartheta^2}{\Gamma^2(\alpha)}\right)^{\frac{1}{2\alpha-1}}, \ \alpha > \frac{1}{2}.$$

Proof. Given that $\sup_{t\in[a,b]} |\theta(t)|^2 \leq c_1$, then we have:

$$E|\varphi(t)|^2 \leq c_1 + \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)} \int_a^t (t-s)^{2\alpha-2} e^{-2(t-s)} \boldsymbol{E}|\varphi(s)|^2 ds.$$

Multiply through by e^{2t} and let $f(t) = e^{2t}E|\varphi(t)|^2$ to obtain:

$$f(t) \leq e^{2t}c_1 + \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)} \int_a^t (t-s)^{2\alpha-2} f(s)ds$$
$$\leq e^{2b}c_1 + \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)} \int_a^t (t-s)^{2\alpha-2} f(s)ds$$
$$= c_3 + \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)} \int_a^t (t-s)^{2\alpha-2} f(s)ds.$$

Then by Proposition 3.10 for $\rho = 2\alpha - 1 > 0$ and $\epsilon = \frac{\delta^2 Lip_\vartheta^2}{\Gamma^2(\alpha)}$, we have:

$$f(t) \leq c_4 exp\left(c_5 \Gamma^{\frac{1}{2\alpha-1}}(2\alpha-1) \frac{\delta^{\frac{2}{2\alpha-1}} Lip_\vartheta^{\frac{2}{2\alpha-1}}}{\Gamma^{\frac{2}{2\alpha-1}}(\alpha)}(t-a)\right), t > a, \quad (4)$$

and the result follows:

Remark 3.12: The best growth bound estimate obtained is the second moment bound. Now, let $0 < q < 1$, and the following elementary inequality holds:

$$(a+b)^q < a^q + b^q \quad (5)$$

Thus, following the proof of Lemma 3.7,

$$E|\varphi(t)|^p \leq c_p + 2^{p-1}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^p \left[\int_a^t (t-s)^{2\alpha-2} e^{-2(t-s)} \boldsymbol{E}|\varphi(s)|^2 ds\right]^{\frac{p}{2}}.$$

Raise both sides to the power of $\frac{2}{p} \leq 1$ and apply Eq. 5 to obtain:

$$(E|\varphi(t)|^p)^{\frac{2}{p}} \leq c_p^{\frac{2}{p}} + 2^{2\left(1-\frac{1}{p}\right)}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^2 \int_a^t (t-s)^{2\alpha-2} e^{-2(t-s)} E|\varphi(s)|^2 ds.$$

It follows that:

$$E|\varphi(t)|^2 \leq (E|\varphi(t)|^p)^{\frac{2}{p}} \leq c_p^{\frac{2}{p}} + 2^{2\left(1-\frac{1}{p}\right)}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^2 \int_a^t (t-s)^{2\alpha-2} e^{-2(t-s)} E|\varphi(s)|^2 ds.$$

Therefore, by following the proof of Theorem 3.11, one gets:

$$f(t) \leq c_p^{\frac{2}{p}} e^{2b} + 2^{2\left(1-\frac{1}{p}\right)}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^2 \int_a^t (t-s)^{2\alpha-2} f(s)ds,$$

and a similar estimate in Eq. 4 easily follows. Therefore,

$$f(t) \leq \tilde{c}_4 exp\left(\tilde{c}_5 \Gamma^{\frac{1}{2\alpha-1}}(2\alpha-1) 2^{\frac{2}{2\alpha-1}\left(1-\frac{1}{p}\right)}\left(\frac{\delta z_p Lip_\vartheta}{\Gamma(\alpha)}\right)^{\frac{2}{2\alpha-1}}(t-a)\right), t > a,$$

for some positive constants $\tilde{c}_4, \tilde{c}_5$; and consequently, we have:

$$E|\varphi(t)|^2 \leq \tilde{c}_4 exp\left(\tilde{c}_6 \delta^{\frac{2}{2\alpha-1}}(t-a) - 2t\right),$$

where,

$$\tilde{c}_6 = \tilde{c}_5 2^{\frac{2}{2\alpha-1}\left(1-\frac{1}{p}\right)}\left(\frac{\Gamma(2\alpha-1) z_p{}^2 Lip_\vartheta^2}{\Gamma^2(\alpha)}\right)^{\frac{1}{2\alpha-1}} > 0$$

3.1.2. Lower moment bound

For the lower growth bound, we use the converse of Proposition 3.10.

Proposition 3.13: (Foondun and Khoshnevisan, 2009) Let $\rho > 0$ and suppose that $f(t)$ is a nonnegative locally integrable function satisfying.

$$f(t) \geq c_1 + \epsilon \int_0^t (t-s)^{\rho-1} f(s)ds, \ \forall \ t > 0,$$

where, $c_1 > 0$. Then we have:

$$f(t) \geq c_2 exp\left(c_3(\Gamma(\rho))^{\frac{1}{\rho}} \epsilon^{\frac{1}{\rho}} t\right),$$

for all

$t > \frac{e}{\rho}(\Gamma(\rho)\epsilon)^{-\frac{1}{\rho}}$

for some positive numbers c_2 and c_3.
Instead of Condition 3.1, we have the following:

Condition 3.14: Let $0 < L_\vartheta < \infty$. Then for all $x \in \mathbb{R}$ and $t \in [a, b]$, we have:

$|\vartheta(t, x)| \leq L_\vartheta |x|$.

Thus, we have the lower bound estimate by assuming that the function $\theta(t) > c_7$ for $c_7 > 0$ to obtain:

Theorem 3.15: Given that Condition 3.1 holds. Then for all $t \in [a, b]$ we obtain:

$E|\varphi(t)|^2 \geq c_8 exp\left(c_{10}\delta^{\frac{2}{2\alpha-1}}(t-a) - 2t\right)$,

for some positive constants c_8, c_9 and,

$c_{10} = c_9\left(\frac{\Gamma(2\alpha-1)L_\vartheta^2}{\Gamma^2(\alpha)}\right)^{\frac{1}{2\alpha-1}}$, $\alpha > \frac{1}{2}$.

Proof. Assume that $\theta(t)$ is bounded below, then by Ito isometry, we have:

$E|\varphi(t)|^2 \geq |\theta(t)|^2 + \frac{\delta^2}{\Gamma^2(\alpha)}\int_a^t (t - s)^{2\alpha-2} e^{-2(t-s)} E\left|\vartheta(s, \varphi(s))\right|^2 ds \geq c_7 + \frac{\delta^2 L_\vartheta^2}{\Gamma^2(\alpha)}\int_a^t (t - s)^{2\alpha-2} e^{-2(t-s)} E|\varphi(s)|^2 ds.$

Let $f(t) = e^{2t}E|\varphi(t)|^2$

to obtain:

$f(t) \geq c_7 + \frac{\delta^2 L_\vartheta^2}{\Gamma^2(\alpha)}\int_a^t (t-s)^{2\alpha-2} f(s) ds.$

Apply Proposition 3.13 for $\rho = 2\alpha - 1 > 0$ and $\epsilon = \frac{\delta^2 L_\vartheta^2}{\Gamma^2(\alpha)}$ to obtain the required estimate.

3.2. Proof of results for Eq. 2

Theorem 3.16: Let $\alpha > \frac{1}{2}$ and suppose Condition 3.2 holds. Then for some positive constants c_2, δ, M such that $c_2 < \frac{1}{(\delta M)^2}$, there exists a unique solution to Eq. 2.

Define the operator:

$\mathfrak{B}\varphi(t) = \theta(t) + \frac{\delta}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} k(t, s, \varphi(s)) dw(s).$

The fixed point of the operator $\mathfrak{B}$ gives the solution of Eq. 2.

Lemma 3.17: Let φ be a mild solution of Eq. 2 satisfying $||\varphi||_2 < \infty$. Suppose Condition 3.2 holds, then for $\alpha > \frac{1}{2}$,

$||\mathfrak{B}\varphi||_2^2 \leq c_1 + c_2\delta^2 M^2 ||\varphi||_2^2$,

with, $c_2 := \frac{(b-a)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}$.

Proof. Take second moment of both sides with Condition 3.2 to obtain:

$E|\mathfrak{B}\varphi(t)|^2 \leq |\theta(t)|^2 + \frac{\delta^2}{\Gamma^2(\alpha)}\int_a^t (t - s)^{2\alpha-2} \boldsymbol{E}|k(t, s, \varphi(s))|^2 ds$
$\leq |\theta(t)|^2 + \frac{\delta^2 M^2}{\Gamma^2(\alpha)}\int_a^t (t-s)^{2\alpha-2} \boldsymbol{E}|\varphi(s)|^2 ds$
$\leq |\theta(t)|^2 + \frac{\delta^2 M^2}{\Gamma^2(\alpha)}||\varphi||_2^2 \int_a^t (t-s)^{2\alpha-2} ds$
$\leq |\theta(t)|^2 + \frac{\delta^2 M^2}{\Gamma^2(\alpha)}||\varphi||_2^2 \frac{(t-a)^{2\alpha-1}}{2\alpha-1}, \qquad \Re(\alpha) > \frac{1}{2}.$

Taking supremum over $t \in [a, b]$ one obtains:

$||\mathfrak{B}\varphi||_2^2 \leq c_1 + \frac{\delta^2 M^2}{(2\alpha-1)\Gamma^2(\alpha)}(b-a)^{2\alpha-1}||\varphi||_2^2$,

and the result follows immediately.

Lemma 3.18: Let φ and ϕ be mild solutions of Eq. 2 satisfying $||\varphi||_2 + ||\phi||_2 < \infty$. Suppose Condition 3.2 holds, then for $\alpha > \frac{1}{2}$,

$||\mathfrak{B}\varphi - \mathfrak{B}\phi||_2^2 \leq c_2\delta^2 M^2 ||\varphi - \phi||_2^2$.

Remark 3.19: The proof of Theorem 3.16 follows readily as the proof of Theorem 3.3.

Remark 3.20: Extending the results in this section to all $p \geq 2$, we state (without proof) the following results:

Lemma 3.21: Let $p \geq 2$, and φ a mild solution of Eq. 2 such that $||\varphi||_p < \infty$. Suppose Condition 3.2 holds, then for $\alpha > \frac{1}{2}$,

$||\mathfrak{B}\varphi||_p^p \leq c_p + \tilde{c}_{\alpha,p}||\varphi||_p^p$,

where, $c_p := 2^{p-1}c$ and,

$\tilde{c}_{\alpha,p} := 2^{p-1}\left(\frac{\delta\, z_p M}{\Gamma(\alpha)}\right)^p \frac{(b-a)^{p(\alpha-\frac{1}{2})}}{(2\alpha-1)^{\frac{p}{2}}}$.

Lemma 3.22: For $p \geq 2$, let φ and ϕ be mild solutions of Eq. 2 satisfying $||\varphi||_p + ||\phi||_p < \infty$. Suppose Condition 3.2 holds, then for $\alpha > \frac{1}{2}$,

$||\mathfrak{B}\varphi - \mathfrak{B}\phi||_p^p \leq \tilde{c}_{\alpha,p}||\varphi - \phi||_p^p$.

and the existence and uniqueness theorem:

Theorem 3.23: Let $\alpha > \frac{1}{2}$ and suppose Condition 3.2 holds. Then for a positive constant $\tilde{c}_{\alpha,p}$ such that $\tilde{c}_{\alpha,p} < 1$, there exists a unique solution to Eq. 2.

3.2.1. Upper moment growth bound

Next, we give the upper moment growth bound by assuming that the function $\theta(t)$ is bounded above:

Theorem 3.24: Given that Condition 3.2 holds. Then for all $t \in [a,b]$ we obtain:

$$E|\varphi(t)|^2 \leq c_{11} exp\left(c_{13}\delta^{\frac{2}{2\alpha-1}}(t-a)\right)$$

for some positive constants c_{11}, c_{12} and,

$$c_{13} = c_{12}\left(\frac{\Gamma(2\alpha-1)M^2}{\Gamma^2(\alpha)}\right)^{\frac{1}{2\alpha-1}}, \ \alpha > \frac{1}{2}.$$

Proof. Let $|\theta(t)|^2 \leq c_1$ for $t \in [a,b]$, then we have:

$$E|\varphi(t)|^2 \leq c_1 + \frac{\delta^2 M^2}{\Gamma^2(\alpha)}\int_a^t (t-s)^{2\alpha-2}\boldsymbol{E}|\varphi(s)|^2 ds.$$

Let $g(t) := E|\varphi(t)|^2$,

to obtain:

$$g(t) \leq c_1 + \frac{\delta^2 M^2}{\Gamma^2(\alpha)}\int_a^t (t-s)^{2\alpha-2} g(s) ds.$$

Thus, the result follows by Proposition 3.10 for $\rho = 2\alpha - 1 > 0$ and $\epsilon = \frac{\delta^2 M^2}{\Gamma^2(\alpha)}$.

3.2.2. Lower moment bound

Now, instead of Condition 3.2, we use:

Condition 3.25: Suppose $0 < m < \infty$. Then for all $x \in \mathbb{R}$ and $t, s \in [a,b]$,

$$|k(t,s,x)| \geq m|x|.$$

Suppose the function $\theta(t) > c_{14}$ for $c_{14} > 0$:

Theorem 3.24: Given that Condition 3.2 holds. Then for some $t \in [a,b]$ we have:

$$E|\varphi(t)|^2 \geq c_{15} exp\left(c_{17}\delta^{\frac{2}{2\alpha-1}}(t-a)\right),$$

for some positive constants c_{15}, c_{16} and,

$$c_{17} = c_{16}\left(\frac{\Gamma(2\alpha-1)m^2}{\Gamma^2(\alpha)}\right)^{\frac{1}{2\alpha-1}}, \ \alpha > \frac{1}{2}.$$

Proof. Since $\theta(t)$ is bounded below, then by Ito isometry, we have:

$$E|\varphi(t)|^2 \geq |\theta(t)|^2 + \frac{\delta^2}{\Gamma^2(\alpha)}\int_a^t (t-s)^{2\alpha-2} E\left|k\big(t, s\varphi(s)\big)\right|^2 ds$$
$$\geq c_{14} + \frac{\delta^2 m^2}{\Gamma^2(\alpha)}\int_a^t (t-s)^{2\alpha-2} E|\varphi(s)|^2 ds.$$

Let $g(t) := E|\varphi(t)|^2$,

to obtain:

$$g(t) \geq c_{14} + \frac{\delta^2 m^2}{\Gamma^2(\alpha)}\int_a^t (t-s)^{2\alpha-2} g(s) ds,$$

and the result readily follows by Proposition 3.13 for $\rho = 2\alpha - 1 > 0$ and $\epsilon = \frac{\delta^2 m^2}{\Gamma^2(\alpha)}$.

4. Conclusion

We studied some stochastic nonlinear fractional Volterra integral equations. The existence and uniqueness of results were given under some continuity conditions on σ and k. The second moment upper and lower growth bounds were obtained, and their exponential growth bounds compared. Further research is to study some asymptotic behaviors of the solutions.

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Compliance with ethical standards

Conflict of interest

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