Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

Fractional formulation of Podolsky Lagrangian density

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ARTICLE INFO

Article history:
Received 25 September 2021
Received in revised form
18 December 2021
Accepted 21 December 2021

Keywords:
Podolsky's Lagrangian density
Fractional derivatives
Lagrangian formulation
Hamiltonian formulation

ABSTRACT

Lagrangians which depend on higher-order derivatives appear frequently in many areas of physics. In this paper, we reformulate Podolsky's Lagrangian in fractional form using left-right Riemann-Liouville fractional derivatives. The equations of motion are obtained using the fractional Euler Lagrange equation. In addition, the energy stress tensor and the Hamiltonian are obtained in fractional form from the Lagrangian density. The resulting equations are very similar to those found in classical field theory.

1. Introduction

Fractional calculus is a branch of mathematics that deals with fractional derivatives and integrals of any order. In the last two decades, it has been increasingly popular in a variety of sectors of science and engineering (Alawaideh et al., 2020; Al-Oqali et al., 2016; Herzallah and Baleanu, 2014; Herzallah et al., 2011; Jaradat, 2017; Muslih and Baleanu, 2005). Fractional derivatives have been used in a variety of fields, including classical mechanics (Yu and Wang, 2017), scaling phenomena (Cattani et al., 2014), fractal spacetime (He, 2014), dispersion and turbulence (Chen et al., 2013), astrophysics (Abdel-Salam et al., 2020), potential theory (Bogdan and Byczkowski, 2000), viscoelasticity (Novikov and Voitsekhovskii, 2000), electrodynamics (La Nave et al., 2019), optics (Asjad et al., 2021; Gutiérrez-Vega, 2007a; 2007b), and thermodynamics (Magomedov et al., 2018). Fractional derivatives research dates back to Leibniz, and it is still going strong today.

Higher derivative field theories have been gaining popularity in recent years. Many models, including renormalizable quantum gravity, Podolsky's generalized electrodynamics, the Lee–Wick model, and others, include higher derivative field equations. Higher derivative theories are being studied for a variety of reasons, including improving renormalization qualities and removing ultraviolet divergences (Kruglov, 2010; Dai, 2021). Podolsky's theory was introduced in the early 1940s by Bopp and Podolsky (Lazar and Leck, 2020). In order to avoid singularities in electromagnetic fields and to have a finite and positive self-energy of point charges, Bopp and Podolsky proposed a gradient theory representing a classical generalization of Maxwell electrodynamics towards generalized electrodynamics with fourth-order linear field equations (Lazar, 2019).

The purpose of this research is to reformulate the Lagrangian, proposed by Podolsky, in fractional form and to obtain conjugate momenta and energy stress tensor.

The following sections of the manuscript are organized as follows: In the next section, we briefly define the Riemann-Liouville fractional derivative. The fractional Euler-Lagrange equations are obtained in section three. In section four, the energy-momentum tensor is constructed, and the Hamiltonian is obtained. The fifth section contains the conclusions.

2. Riemann-Liouville fractional derivative

The left and right Riemann-Liouville fractional derivatives are defined as follows (Diab et al., 2013):

- The left Riemann-Liouville fractional derivative,

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1}\, f(\tau)d\tau \qquad (1)$$

- The right Riemann-Liouville fractional derivative,

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_a^t (\tau-t)^{n-\alpha-1}\, f(\tau)d\tau \qquad (2)$$

where α represents the order of the derivative such that $n-1 \le \alpha < n$ and Γ represents the Euler's

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https://doi.org/10.21833/ijaas.2022.02.015
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Lagrange gamma function. If α is an integer, these derivatives are defined in the usual sense, i.e.,

$$ {}_{a}D_{t}^{\alpha}f(t) = \left(\frac{d}{dt}\right)^{\alpha}, \qquad {}_{t}D_{b}^{\alpha}f(t) = \left(-\frac{d}{dt}\right)^{\alpha}, \alpha = 1,2,... \quad (3)$$

3. Formulation

Assume that the Lagrangian L is a function of the potential A_α and its first and second derivatives:

$$L = L\left(A_\alpha, \ {}_bD_{x_\alpha}^{\beta} A_\alpha, \ {}_bD_{x_\alpha}^{\beta} \ {}_bD_{x_\alpha}^{\gamma} A_\alpha\right) \quad (4)$$

where A_α are functions of space-time coordinates $x_\alpha = x_1, x_2, x_3, x_4$. The variational equation:

$\delta W = \delta \iint LV\, dt = 0, dV = dx_1\, d\, x_2\, d\, x_3$,
or $\delta W = \delta \iint L d\Omega = 0, \ \ d\Omega = dV\, d\, x_4 = 0$ (5)

this results in the field equation.

$$\frac{\partial \mathcal{L}}{\partial A_\lambda} - \ {}_bD_{x_\mu}^{\rho}\left(\frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\mu}^{\varkappa} A_\lambda\right)}\right) + \ {}_bD_{x_\mu}^{\varkappa}\, {}_bD_{x_\eta}^{\rho} \frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\mu}^{\varkappa}\, {}_bD_{x_\eta}^{\rho} A_\lambda\right)} = 0 \quad (6)$$

as new coordinates, we introduce:

$$q_\alpha = A_\alpha \text{ and } Q_\alpha = \ {}_aD_{x_0}^{\beta} A_\alpha \quad (7)$$

and define the momenta conjugate to q_α and Q_α by,

$$p_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\beta}^{\rho} A_\alpha\right)} - \ {}_bD_{x_\mu}^{k}\left(\frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\mu}^{k} \ {}_bD_{x_\beta}^{\rho} A_\alpha\right)}\right) \quad (8)$$

and,

$$P_\alpha = \frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_0}^{\beta} \ {}_bD_{x_0}^{\beta} A_\alpha\right)} \quad (9)$$

respectively. The Hamiltonian can be described as follows:

$$H = -L + p_{\alpha\beta} \ {}_bD_{x_0}^{\beta} A_\alpha + \frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_0}^{\beta} \ {}_bD_{x_0}^{\rho} A_\alpha\right)} \ {}_bD_{x_0}^{\beta} \ {}_bD_{x_0}^{\beta} A_\alpha \quad (10)$$

Using Eqs. 7 and 9 the time derivatives of the coordinates, ${}_aD_{x_0}^{\beta} A_\alpha$ and ${}_bD_{x_0}^{\beta} \ {}_bD_{x_0}^{\beta} A_\alpha$, can be removed from the Hamiltonian. Then,

$$H = H\left(A_\alpha, p_{\alpha\beta}, \ {}_bD_{x_i}^{\alpha} A_\alpha, \ {}_bD_{x_i}^{\beta} \ {}_bD_{x_j}^{\beta} A_\alpha, \ {}_bD_{x_0}^{\beta} A_\alpha, P_\alpha, \ {}_bD_{x_i}^{\beta}\, {}_bD_{x_0}^{\beta} A_\alpha\right). \quad (11)$$

Taking the differentials of Eqs. 10 and 11 and equating coefficients we get:

$$\frac{\partial H}{\partial\left(\ {}_bD_{x_0}^{\beta} A_\alpha\right)} = -\frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\mu}^{\beta} A_\alpha\right)} + \frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\beta}^{\rho} A_\alpha\right)} - \ {}_bD_{x_\mu}^{k}\left(\frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\mu}^{k} \ {}_bD_{x_\beta}^{\rho} A_\alpha\right)}\right) \quad (12a)$$

$$\frac{\partial H}{\partial\left(\ {}_bD_{x_i}^{\alpha} A_\alpha\right)} = -\frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_i}^{\alpha} A_\alpha\right)} \quad (12b)$$

$$\frac{\partial H}{\partial\left(\ {}_bD_{x_i}^{\alpha} A_\alpha\right)} = -\frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_i}^{\alpha} A_\alpha\right)} \quad (12c)$$

$$\frac{\partial H}{\partial\left(\ {}_bD_{x_i}^{\alpha} \ {}_bD_{x_j}^{\beta} A_\alpha\right)} = -\frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_i}^{\alpha} \ {}_bD_{x_j}^{\beta} A_\alpha\right)} \quad (12d)$$

and,

$$\frac{\partial H}{\partial (p_\alpha)} = \ {}_bD_{x_0}^{\beta} A_\alpha, \frac{\partial H}{\partial (P_\alpha)} = \ {}_bD_{x_0}^{\beta} \ {}_bD_{x_0}^{\beta} A_\alpha \quad (13)$$

It may now be concluding from Eqs. 8, 9, 12, 13, and 6 that:

$$\frac{\partial H}{\partial\left(\ {}_bD_{x_i}^{\alpha} A_\alpha\right)} = -{}_bD_{x_0}^{\beta} P - \ {}_bD_{x_i}^{\alpha} \frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_i}^{\beta} A_\alpha\right)} \quad (14)$$

$${}_bD_{x_0}^{\beta}\, p_\alpha = -\frac{\partial H}{\partial\left(\ {}_bD_{x_i}^{\alpha} A_\alpha\right)} + \ {}_bD_{x_i}^{\alpha} \frac{\partial H}{\partial\left(\ {}_bD_{x_i}^{\alpha} A_\alpha\right)} - \ {}_bD_{x_i}^{\alpha} \ {}_bD_{x_j}^{\alpha} \frac{\partial H}{\partial\left(\ {}_bD_{x_i}^{\alpha} \ {}_bD_{x_j}^{\beta} A_\alpha\right)} \quad (15)$$

$${}_bD_{x_0}^{\beta} P = -\frac{\partial H}{\partial\left(\ {}_bD_{x_i}^{\alpha} A_\alpha\right)} + \ {}_bD_{x_i}^{\alpha} \frac{\partial H}{\partial\left(\ {}_bD_{x_i}^{\alpha} \ {}_bD_{x_0}^{\beta} A_\alpha\right)} \quad (16)$$

The variational equation can be used to calculate the energy-momentum stress tensor.

$$\delta W = \iint \left(L\delta_{\mu\nu} - \ {}_bD_{x_\alpha}^{\alpha} \ {}_bD_{x_\mu}^{\alpha} A_\alpha p_{\alpha\nu} - \ {}_bD_{x_\alpha}^{\alpha} \ {}_bD_{x_\mu}^{\alpha} \ {}_bD_{x_\lambda}^{\alpha} A_\alpha P_{\alpha\lambda\nu}\right) \boldsymbol{dS_\nu \delta x_\nu} \quad (17)$$

where,

$$p_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\beta}^{\varkappa} A_\alpha\right)} - \ {}_bD_{x_\mu}^{\varkappa} \frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\beta}^{\varkappa}\, {}_bD_{x_\mu}^{\rho} A_\alpha\right)} \quad (18)$$

$$P_{\alpha\beta\gamma} = \frac{\partial \mathcal{L}}{\partial\left(\ {}_bD_{x_\beta}^{\varkappa}\, {}_bD_{x_\mu}^{\rho} A_\alpha\right)}. \quad (19)$$

From the definition of the total momentum P_μ,

$$\delta W = P_\mu \delta x_\mu. \quad (20)$$

By comparison with Eq. 17:

$$P_\mu = \int T_{\mu\nu} dS_\nu \quad (21)$$

where the energy-momentum tensor $T_{\mu\nu}$ is given by,

$$T_{\mu\nu} = L\delta_{\mu\nu} - \left(\ {}_bD_{x_\mu}^{k} A_\alpha\right) P_{\alpha\nu} - \left(\ {}_bD_{x_\mu}^{k} \ {}_bD_{x_\lambda}^{n} A_\alpha\right) P_{\alpha\lambda\nu} \quad (22)$$

4. Fractional Lagrangian density

Let us consider the Lagrangian proposed by Podolsky (Bertin et al., 2017; Podolsky and Schwed, 1948):

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{a^2}{2} \ {}_bD_{x_\beta}^{\varkappa} F^{\alpha\beta}\, {}_bD_{x_\gamma}^{\rho} F_{\alpha\gamma} \quad (23)$$

where the field quantities $F_{\alpha\beta} = \ {}_bD_{x_\alpha}^{\varkappa} A_\beta - \ {}_bD_{x_\beta}^{\varkappa} A_\alpha$, and a being a parameter with a dimension of the inverse of mass.

Using the definition of the left Riemann-Liouville fractional derivative, the fractional Lagrangian density takes the form:

$$\mathcal{L} = -\frac{1}{4} g^{\alpha\sigma} g^{\beta\nu}\left[\ {}_bD_{x_\alpha}^{\varkappa} A_\beta - \ {}_bD_{x_\beta}^{\varkappa} A_\alpha\right]\left[\ {}_bD_{x_\sigma}^{\varkappa} A_\nu - \ {}_bD_{x_\nu}^{\varkappa} A_\sigma\right] - \frac{a^2}{2} g^{\alpha\sigma} g^{\beta\nu}\left[\ {}_bD_{x_\beta}^{\varkappa}\, {}_bD_{x_\sigma}^{\rho} A_\nu - \ {}_bD_{x_\beta}^{\varkappa}\, {}_bD_{x_\nu}^{\rho} A_\sigma\right] g^{\gamma\xi}\left[\ {}_bD_{x_\xi}^{\varkappa}\, {}_bD_{x_\alpha}^{\rho} A_\gamma - \ {}_bD_{x_\xi}^{\varkappa}\, {}_bD_{x_\gamma}^{\rho} A_\alpha\right]. \quad (24)$$

But ${}_bD^{\rho}_{x_\nu} = ({}_bD^{\rho}_{x_t}, {}_bD^{\rho}_{x_j})$. The Euler Lagrange equation is given by,

$$\frac{\partial \mathcal{L}}{\partial A_\lambda} - {}_bD^{\rho}_{x_\mu}\left(\frac{\partial \mathcal{L}}{\partial({}_bD^{\varkappa}_{x_\mu}A_\lambda)}\right) + {}_bD^{\varkappa}_{x_\mu}\,{}_bD^{\rho}_{x_\eta}\frac{\partial \mathcal{L}}{\partial({}_bD^{\varkappa}_{x_\mu}\,{}_bD^{\rho}_{x_\eta}A_\lambda)} = 0. \quad (25)$$

First term:

$$\frac{\partial \mathcal{L}}{\partial A_\lambda} = 0. \quad (26)$$

Second term:

$${}_bD^{\rho}_{x_\mu}\left(\frac{\partial \mathcal{L}}{\partial({}_bD^{\varkappa}_{x_\mu}A_\lambda)}\right) = -\frac{1}{4}g^{\alpha\sigma}g^{\beta\nu}\left[\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\lambda} - \delta^{\beta}_{\mu}\delta^{\alpha}_{\lambda}\right)\left({}_bD^{\varkappa}_{x_\sigma}A_\nu - {}_bD^{\varkappa}_{x_\nu}A_\sigma\right) + \left({}_bD^{\varkappa}_{x_\alpha}A_\beta - {}_bD^{\varkappa}_{x_\beta}A_\alpha\right)\left(\delta^{\alpha}_{\mu}\delta^{\nu}_{\lambda} - \delta^{\nu}_{\mu}\delta^{\sigma}_{\lambda}\right)\right] \quad (27)$$

or,

$${}_bD^{\rho}_{x_\mu}\left(\frac{\partial \mathcal{L}}{\partial({}_bD^{\varkappa}_{x_\mu}A_\lambda)}\right) = -\left({}_bD^{\varkappa}_{x^\mu}A^\lambda - {}_bD^{\varkappa}_{x^\lambda}A^\mu\right) = -F^{\mu\lambda} \quad (28)$$

Then,

$${}_bD^{\rho}_{x_\mu}\left(\frac{\partial \mathcal{L}}{\partial({}_bD^{\varkappa}_{x_\mu}A_\lambda)}\right) = -{}_bD^{\rho}_{x_\mu}F^{\mu\lambda}. \quad (29)$$

Third term:

$${}_bD^{\varkappa}_{x_\mu}\,{}_bD^{\rho}_{x_\eta}\frac{\partial \mathcal{L}}{\partial({}_bD^{\varkappa}_{x_\mu}\,{}_bD^{\rho}_{x_\eta}A_\lambda)} = -\frac{a^2}{2}\,{}_bD^{\varkappa}_{x_\mu}\,{}_bD^{\rho}_{x_\eta}\left[g^{\alpha\sigma}g^{\beta\nu}\left(\delta^{\beta}_{\mu}\delta^{\sigma}_{\eta}\delta^{\nu}_{\lambda} - \delta^{\beta}_{\mu}\delta^{\nu}_{\lambda}\delta^{\alpha}_{\lambda}\right){}_bD^{\varkappa}_{x^\gamma}F_{\alpha\gamma} + {}_bD^{\omega}_{x_\beta}F^{\alpha\beta}g^{\gamma\xi}\left(\delta^{\xi}_{\mu}\delta^{\alpha}_{\eta}\delta^{\gamma}_{\lambda} - \delta^{\xi}_{\mu}\delta^{\gamma}_{\eta}\delta^{\alpha}_{\lambda}\right)\right] \quad (30)$$

$$= -a^2\left[{}_bD^{\rho}_{x_\eta}\,{}_bD^{\kappa}_{x^\lambda}\,{}_bD^{\omega}_{x_\beta}F^{\eta\beta} - \Box\,{}_bD^{\omega}_{x_\beta}F^{\lambda\beta}\right]. \quad (31)$$

Using the identity, see Appendix A,

$${}_bD^{\rho}_{x_\eta}\,{}_bD^{\omega}_{x_\beta}F^{\eta\beta} = 0.$$

Therefore,

$${}_bD^{\varkappa}_{x_\mu}\,{}_bD^{\rho}_{x_\eta}\frac{\partial \mathcal{L}}{\partial({}_bD^{\varkappa}_{x_\mu}\,{}_bD^{\rho}_{x_\eta}A_\lambda)} = a^2\Box\, D^{k}_{x_\beta}F^{\lambda\beta}. \quad (32)$$

Substituting Eqs. 26, 29, and 32 in Eq. 25, we get

$$D^{\rho}_{x_\mu}F^{\mu\lambda} + a^2\Box\,{}_bD^{k}_{x_\beta}F^{\lambda\beta} = 0 \quad (33)$$

change β to μ we obtain,

$$(1 - a^2\Box)\;{}_bD^{k}_{x_\mu}F^{\mu\lambda} = 0 \quad (34)$$

5. Energy-momentum tensor

To evaluate the energy-momentum tensor the following quantities $p_{\alpha\beta}$ and $P_{\mu\lambda\beta}$ are required

$$p_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial({}_bD^{\rho}_{x_\beta}A_\alpha)} - {}_bD^{k}_{x_\mu}\left(\frac{\partial \mathcal{L}}{\partial({}_bD^{k}_{x_\mu}\,{}_bD^{\rho}_{x_\beta}A_\alpha)}\right)$$

$$p_{\alpha\beta} = -F^{\beta\alpha} - {}_aD^{k}_{x_\mu}\left[\frac{-a^2}{2}g^{\grave{\alpha}\sigma}g^{\grave{\beta}\nu}\left(\delta^{\mu}_{\sigma}\delta^{\beta}_{\grave{\beta}}\delta^{\alpha}_{\nu} - \delta^{\beta}_{\grave{\beta}}\delta^{\mu}_{\nu}\delta^{\alpha}_{\sigma}\right)\left({}_bD^{\omega}_{x^\gamma}\,{}_bD^{\omega}_{x_\alpha}A_\gamma - {}_bD^{\omega}_{x^\gamma}\,{}_bD^{\omega}_{x^\gamma}A_\alpha\right)\frac{a^2}{2}g^{\grave{\alpha}\sigma}g^{\grave{\beta}\nu}\left({}_bD^{\rho}_{x_\beta}\,{}_bD^{l}_{x_\sigma}A_\nu - {}_bD^{\rho}_{x_{\grave{\beta}}}\,{}_bD^{m}_{x_\nu}A_\sigma\right)g^{\gamma\xi}\left(\delta^{\mu}_{\xi}\delta^{\beta}_{\grave{\alpha}}\delta^{\alpha}_{\gamma} - \delta^{\mu}_{\xi}\delta^{\beta}_{\gamma}\delta^{\alpha}_{\grave{\alpha}}\right)\right] \quad (35)$$

we are able to recast the above equation as follows:

$$p_{\alpha\beta} = -F^{\beta\alpha} + a^2\;{}_bD^{k}_{x_\mu}\left[g^{\alpha\mu}\;{}_bD^{\omega}_{x^\gamma}F^{\beta}{}_{\gamma} - g^{\beta\mu}\;{}_bD^{\omega}_{x^\gamma}F^{\alpha}{}_{\gamma}\right]$$
$$= -F^{\beta\alpha} + a^2\;{}_bD^{\omega}_{x_\gamma}\left[{}_bD^{k}_{x^\alpha}F^{\beta\gamma} - {}_bD^{\rho}_{x^\beta}F^{\alpha\gamma}\right]$$
$$= -F^{\beta\alpha} + a^2\;{}_bD^{\omega}_{x_\gamma}\left[{}_bD^{\omega}_{x^\gamma}\left({}_bD^{\rho}_{x^\beta}A^\alpha - {}_bD^{k}_{x^\alpha}A^\beta\right)\right]$$
$$= -F^{\beta\alpha} + a^2\;{}_bD^{\omega}_{x_\nu}\;{}_bD^{\omega}_{x^\nu}F^{\beta\alpha}$$
$$= (1 - a^2\Box)F^{\alpha\beta} \quad (36)$$

on the other hand, the total momentum can be obtained as follows:

$$P_{\mu\lambda\beta} = \frac{\partial \mathcal{L}}{\partial({}_bD^{n}_{x_\lambda}\;{}_bD^{\rho}_{x_\beta}A_\mu)} = -\frac{1}{2}a^2g^{\alpha\sigma}g^{\grave{\beta}\nu}\left(\delta^{\lambda}_{\grave{\beta}}\delta^{\beta}_{\sigma}\delta^{\mu}_{\nu} - \delta^{\lambda}_{\grave{\beta}}\delta^{\beta}_{\nu}\delta^{\mu}_{\sigma}\right)\;{}_bD^{\xi}_{x^\gamma}F_{\alpha\gamma} - \frac{1}{2}a^2\;{}_bD^{\rho}_{x_{\grave{\beta}}}F^{\alpha\grave{\beta}}g^{\gamma\xi}\left(\delta^{\lambda}_{\xi}\delta^{\beta}_{\alpha}\delta^{\mu}_{\gamma} - \delta^{\lambda}_{\xi}\delta^{\beta}_{\gamma}\delta^{\mu}_{\alpha}\right) \quad (37)$$

the above equation can be written as:

$$= -\frac{1}{2}a^2\left[\left({}_bD^{\omega}_{x^\gamma}F^{\beta}{}_{\gamma} + {}_bD^{\rho}_{x_{\grave{\beta}}}F^{\beta\grave{\beta}}\right)g^{\lambda\mu} - \left({}_bD^{\omega}_{x_\gamma}F^{\mu\gamma} + {}_bD^{\rho}_{x_{\grave{\beta}}}F^{\mu\grave{\beta}}\right)g^{\beta\lambda}\right]$$

or,

$$P_{\mu\lambda\beta} = -a^2\left[{}_bD^{\omega}_{x_\gamma}F^{\beta\gamma}g^{\lambda\mu} - {}_bD^{\omega}_{x_\gamma}F^{\mu\gamma}g^{\beta\lambda}\right] \quad (38)$$

replacing λ by β in the previous equation we get:

$$P_{\mu\lambda\beta} = -a^2\left[{}_aD^{\omega}_{x_\gamma}F^{\lambda\gamma}g^{\beta\mu} - {}_aD^{\omega}_{x_\gamma}F^{\mu\gamma}g^{\beta\lambda}\right]. \quad (39)$$

Thus the energy-momentum tensor $T_{\mu\nu}$ is given by:

$$T_{\mu\nu} = L\delta_{\mu\nu} - \left({}_bD^{k}_{x_\mu}A_\alpha\right)P_{\alpha\nu} - \left({}_bD^{k}_{x_\mu}\;{}_bD^{n}_{x_\lambda}A_\alpha\right)P_{\alpha\lambda\nu} \quad (40)$$

or,

$$= L\delta_{\mu\nu} - {}_bD^{k}_{x_\mu}A_\alpha(1 - a^2\Box)F^{\alpha\nu} + a^2\;{}_bD^{k}_{x_\mu}\;{}_bD^{n}_{x_\lambda}A_\alpha\left({}_bD^{\omega}_{x_\gamma}F^{\lambda\gamma}g^{\nu\alpha} - {}_bD^{\omega}_{x_\gamma}F^{\alpha\gamma}g^{\nu\lambda}\right)$$
$$= L\delta_{\mu\nu} - {}_bD^{k}_{x_\mu}A_\alpha(1 - a^2\Box)F^{\alpha\nu} + a^2\left[{}_bD^{k}_{x_\mu}\left({}_bD^{n}_{x_\alpha}A^\nu - {}_bD^{m}_{x^\nu}A_\alpha\right)\right]\;{}_bD^{\omega}_{x_\gamma}F^{\alpha\gamma}.$$

The above equation can be simplified to:

$$T_{\mu\nu} = L\delta_{\mu\nu} - {}_bD^{k}_{x_\mu}A_\alpha(1 - a^2\Box)F^{\alpha\nu} + a^2\;{}_bD^{k}_{x_\mu}F^{\nu}{}_{\alpha}\;{}_aD^{\omega}_{x_\gamma}F^{\alpha\gamma}. \quad (41)$$

The Hamiltonian is given by:

$$H = T_{00} = L - {}_bD^k_{x_0}A_\alpha(1 - a^2\Box)F^{\alpha 0} + a^2 \; {}_bD^k_{x_0}F^0_\alpha \; {}_bD^\omega_{x_\gamma}F^{\alpha\gamma}. \quad (42)$$

Because of the ambiguity in the partial differential equation with regard to $\dot{A}_\alpha$. Care must be taken while computing the momenta canonically conjugate to A_α and $\dot{A}_\alpha$.

$$H = L - {}_bD^k_{x_0}A_\alpha(1 - a^2\Box)F^{\alpha 0} + a^2 \; {}_bD^k_{x_0}F_{\alpha 0} \; {}_bD^\omega_{x_\gamma}F^{\alpha\gamma}. \quad (43)$$

We have from Eq. 18:

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial({}_bD^k_{x_0}A_\alpha)} - {}_bD^k_{x_0}\left(\frac{\partial \mathcal{L}}{\partial({}_bD^k_{x_0}\,{}_bD^k_{x_0}A_\alpha)}\right) - {}_bD^k_{x_j}\left(\frac{\partial \mathcal{L}}{\partial({}_bD^k_{x_0}\,{}_bD^k_{x_j}A_\alpha)} + \frac{\partial \mathcal{L}}{\partial({}_bD^k_{x_j}\,{}_bD^k_{x_0}A_\alpha)}\right). \quad (44)$$

Using Eqs. 1, 2, 3, and 4 in Appendix B we get,

$$p_\alpha = F^{\alpha 0} + a^2\left({}_bD^k_{x_j}\;{}_bD^\omega_{x_\gamma}F^{0\gamma}g^{j\alpha} + {}_bD^\rho_{x_\beta}\;{}_bD^\omega_{x_\gamma}F^{\beta\gamma} - {}_bD^k_{x_0}\;{}_bD^\omega_{x_\gamma}F^{\alpha\gamma}\right).$$

After some algebraic manipulations, we are capable of arriving at:

$$p_\alpha = F^{\alpha 0} + a^2\left({}_bD^k_{x_j}\;{}_bD^\omega_{x_\gamma}F^{0\gamma}g^{j\alpha} - {}_bD^k_{x_0}\;{}_bD^\omega_{x_\gamma}F^{\alpha\gamma}\right) \quad (45)$$

thus we obtain:

$$p_0 = F^{00} + a^2\left({}_bD^k_{x_j}\;{}_bD^\omega_{x_\gamma}F^{0\gamma}g^{j0} - {}_bD^k_{x_0}\;{}_bD^\omega_{x_\gamma}F^{0\gamma}\right) \quad (46)$$

$$= -a^2\;{}_bD^k_{x_0}\;{}_bD^\omega_{x_\gamma}F^{0\gamma} = -a^2\;{}_bD^k_{x_0}\;{}_bD^k_{x_0}F^{00} - a^2\;{}_bD^k_{x_0}\;{}_bD^k_{x_j}F^{0j}$$

$$= -a^2\;{}_bD^k_{x_0}\;{}_bD^k_{x_j}F^{0j} = a^2\;{}_bD^k_{x_0}\;{}_bD^k_{x_j}F^{j0} \quad (47)$$

and

$$p_l = F^{l0} + a^2\left({}_bD^k_{x_j}\;{}_bD^\omega_{x_\gamma}F^{0\gamma}g^{jl} - {}_bD^k_{x_0}\;{}_bD^\omega_{x_\gamma}F^{l\gamma}\right) \quad (48)$$

$$= F^{l0} + a^2\left({}_bD^k_{x^l}\;{}_bD^\omega_{x_\gamma}F^{0\gamma} - {}_bD^k_{x_0}\;{}_bD^\omega_{x_\gamma}F^{l\gamma}\right)$$

$$= F^{l0} + a^2\;{}_bD^\omega_{x_\gamma}\left({}_bD^k_{x^l}F^{0\gamma} - {}_bD^k_{x_0}F^{l\gamma}\right) \quad (49)$$

using the identity:

$${}_bD^n_{x^\lambda}F^{\mu\nu} + {}_bD^k_{x^\mu}F^{\nu\lambda} + {}_bD^m_{x^\nu}F^{\lambda\mu} = 0 \quad (50)$$

With $\lambda = l, \mu = 0$, and $\nu = \gamma$

$${}_bD^k_{x^l}F^{0\gamma} + {}_bD^k_{x^0}F^{\gamma l} + {}_bD^\omega_{x^\gamma}F^{l0} = 0 \quad (51)$$

Then,

$${}_bD^k_{x^l}F^{0\gamma} - {}_bD^k_{x^0}F^{l\gamma} = -{}_bD^\omega_{x^\gamma}F^{l0}$$

Eq. 49 becomes:

$$p_l = F^{l0} - a^2\;{}_bD^\omega_{x_\gamma}\;{}_bD^\omega_{x_\gamma}F^{l0} = (1 - a^2\Box)F^{l0} \quad (52)$$

similarly,

$$P_\alpha = \frac{\partial \mathcal{L}}{\partial({}_aD^k_{x_0}\;{}_aD^k_{x_0}A_\alpha)} = -a^2\left({}_aD^\rho_{x_\beta}F^{0\beta}g^{\alpha 0} - {}_aD^\rho_{x_\beta}F^{\alpha\beta}\right) \quad (53)$$

or,

$$P_0 = -a^2\left({}_aD^\rho_{x_\beta}F^{0\beta}g^{00} - {}_aD^\rho_{x_\beta}F^{0\beta}\right) = 0 \quad (54)$$

$$P_j = -a^2\left({}_aD^\rho_{x_\beta}F^{0\beta}g^{j0} - {}_aD^\rho_{x_\beta}F^{j\beta}\right) = -a^2\;{}_aD^\rho_{x_\beta}F^{j\beta}. \quad (55)$$

In vector notation, these quantities become i.e. when ($\alpha \longrightarrow 1$):

$$p_0 = -a^2\;{}_bD^k_{x_0}\;{}_bD^k_{x_j}F^{j0} = \frac{a^2}{c^2}\vec{\nabla}\cdot\vec{E} \quad (56)$$

$$p_j = (1 - a^2)F^{k0} = (1 - a^2\Box)\vec{E} \quad (57)$$

$$P_0 = 0, P_j = a^2\;{}_bD^\rho_{x_\beta}F^{j\beta} \quad (58)$$

$$\vec{P} = a^2\left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right) \quad (59)$$

6. Conclusion

In various areas of physics, Lagrangians depending on higher-order derivatives appear regularly. Using special properties of the higher derivative terms, advances in the understanding of astrophysical and cosmological behaviors were considered feasible. In this work, we have considered the Lagrangian density of higher derivative generalized electrodynamics proposed by Podolsky and obtained the equations of motion, the energy stress tensor, the canonical Momentum, and the Hamiltonian. A major point in this study is that we used the energy stress tensor to evaluate the Hamiltonian of the system. As a special case, for the equation of motion in agreement with the classical results.

Appendix A. Proof of field- strength tensor identity

Prove that ${}_bD^\rho_{x_\eta}\,{}_bD^\omega_{x_\beta}F^{\eta\beta} = 0$, We have:

$${}_bD^\rho_{x_\eta}\,{}_bD^\omega_{x_\beta}F^{\eta\beta} = {}_bD^\rho_{x_\eta}\,{}_bD^\omega_{x_\beta}\left({}_bD^k_{x^\eta}A^\beta - {}_bD^k_{x^\beta}A^\eta\right) = {}_bD^\rho_{x_\beta}\;{}_bD^\omega_{x_\eta}\;{}_bD^k_{x^\eta}A^\beta - {}_bD^\rho_{x_\eta}\;{}_bD^\omega_{x_\beta}\;{}_bD^k_{x^\beta}A^\eta$$

$$= {}_bD^\rho_{x_\beta}\Box A^\beta - {}_bD^\rho_{x_\eta}\Box A^\eta.$$

Change η to β in the second term, we get:

$${}_bD^\rho_{x_\eta}\,{}_bD^\omega_{x_\beta}F^{\eta\beta} = 0. \quad (A1)$$

Appendix B. Calculation of conjugate momenta

We can rewrite Eq. 44 as:

First term:

$$\frac{\partial \mathcal{L}}{\partial({}_bD^x_{x_0}A_\alpha)} = F^{\alpha 0}. \quad (B1)$$

Second term:

$$\frac{\partial \mathcal{L}}{\partial\left({}_bD_{x_0}^k\ {}_bD_{x_0}^k A_\rho\right)} = -\frac{1}{2}a^2 g^{\alpha\sigma} g^{\beta\nu}\left(\delta_\beta^0\delta_\sigma^0\delta_\nu^\rho - \delta_\beta^0\delta_\nu^0\delta_\sigma^\rho\right)\ {}_bD_{x^\gamma}^\omega F_{\alpha\gamma} - \frac{1}{2}a^2\ {}_bD_{x_\beta}^n F^{\alpha\beta} g^{\gamma\xi}\left(\delta_\xi^0\delta_\alpha^0\delta_\gamma^\rho - \delta_\xi^0\delta_\gamma^0\delta_\sigma^\rho\right).$$

After some mathematical manipulation we have:

$$\frac{\partial \mathcal{L}}{\partial\left({}_bD_{x_0}^k\ {}_bD_{x_0}^k A_\rho\right)} = -\frac{1}{2}a^2\left(g^{\alpha 0}g^{0\rho} - g^{\alpha\rho}g^{00}\right)\ {}_bD_{x^\gamma}^\omega F_{\alpha\gamma} + {}_bD_{x_\beta}^n F^{\alpha\beta}\left(g^{\rho 0}\delta_\alpha^0 - g^{00}\delta_\alpha^\rho\right)$$
$$= -\frac{1}{2}a^2\left(g^{\alpha 0}\ {}_bD_{x^\gamma}^\omega F_{0\gamma} - {}_bD_{x^\gamma}^\omega F^\rho{}_\gamma + {}_bD_{x_\beta}^n F^{0\gamma}g^{\rho 0} - {}_bD_{x_\beta}^n F^{\rho\gamma}\right)$$
$$= -a^2\left[\ {}_bD_{x_\gamma}^\omega F^{0\gamma}g^{\rho 0} - {}_bD_{x_\gamma}^\omega F^{\rho\gamma}\right].$$

Then,

$${}_bD_{x_0}^k\left(\frac{\partial \mathcal{L}}{\partial\left({}_bD_{x_0}^k\ {}_bD_{x_0}^k A_\rho\right)}\right) = -a^2\left[\ {}_bD_{x_0}^k\ {}_bD_{x_\gamma}^\omega F^{0\gamma} - {}_bD_{x_0}^k\ {}_bD_{x_\gamma}^\omega F^{\rho\gamma}\right]$$
$${}_bD_{x_0}^k \frac{\partial \mathcal{L}}{\partial\left({}_bD_{x_0}^k\ {}_bD_{x_0}^k A_\alpha\right)} = -a^2\left[\ {}_bD_{x_0}^k\ {}_bD_{x_\gamma}^\omega F^{0\gamma} - {}_bD_{x_0}^k\ {}_bD_{x_\gamma}^\omega F^{\alpha\gamma}\right]. \tag{B2}$$

Third Term:

$$\frac{\partial \mathcal{L}}{\partial\left({}_bD_{x_0}^k\ {}_bD_{x_j}^k A_\alpha\right)} = -\frac{1}{2}a^2 g^{\alpha\sigma} g^{\beta\nu}\left(\delta_\beta^0\delta_\sigma^j\delta_\nu^\rho - \delta_\beta^0\delta_\nu^j\delta_\sigma^\rho\right)\ {}_bD_{x^\gamma}^\omega F_{\alpha\gamma} - \frac{1}{2}a^2\ {}_bD_{x_\beta}^n F^{\alpha\beta} g^{\gamma\xi}\left(\delta_\xi^0\delta_\alpha^j\delta_\gamma^\rho - \delta_\xi^0\delta_\gamma^\nu\delta_\alpha^\rho\right).$$

After some mathematical manipulation, we get,

$$\frac{\partial \mathcal{L}}{\partial\left({}_bD_{x_0}^k\ {}_bD_{x_j}^k A_\alpha\right)} = -\frac{1}{2}a^2\left[\left(g^{\alpha j}g^{0\rho} - g^{\alpha\rho}g^{0j}\right)\ {}_bD_{x^\gamma}^\omega F_{\alpha\gamma} + {}_bD_{x_\beta}^n F^{\alpha\beta}g^{\gamma\xi}\left(g^{0\rho}\delta_\alpha^j - g^{0j}\delta_\alpha^\rho\right)\right]$$
$$\frac{\partial \mathcal{L}}{\partial\left({}_bD_{x_0}^k\ {}_bD_{x_j}^k A_\alpha\right)} = -\frac{1}{2}a^2\left[\ {}_bD_{x^\gamma}^\omega F_\gamma^j g^{0\rho} + {}_bD_{x_\beta}^\rho F^{j\beta}g^{0\rho}\right] = -a^2\ {}_bD_{x_\gamma}^\omega F^{j\gamma}. \tag{B3}$$

Fourth term:

$$\frac{\partial \mathcal{L}}{\partial\left({}_bD_{x_j}^k\ {}_bD_{x_0}^k A_\rho\right)} = -\frac{1}{2}a^2\left[g^{\alpha 0}g^{j\rho}\ {}_bD_{x^\gamma}^\omega \mathrm{F}_{\alpha\gamma} + {}_bD_{x_\beta}^\omega F^{j\beta}g^{j\rho}\delta_\alpha^0\right]$$
$$= -\frac{1}{2}a^2\left[\ {}_bD_{x^\gamma}^\omega \mathrm{F}_{0\gamma}g^{j\rho} + g^{j\rho}\ {}_bD_{x_\beta}^\omega F^{0\beta}\right].$$

It follows that:

$$\frac{\partial \mathcal{L}}{\partial\left({}_bD_{x_j}^k\ {}_bD_{x_0}^k A_\rho\right)} = -a^2\ {}_bD_{x_\beta}^\omega F^{0\beta}\, g^{j\rho}. \tag{B4}$$

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

References

Abdel-Salam EAB, Nouh MI, and Elkholy EA (2020). Analytical solution to the conformable fractional Lane-Emden type equations arising in astrophysics. Scientific African, 8: e00386. https://doi.org/10.1016/j.sciaf.2020.e00386

Alawaideh Y M, Hijjawi RS, and Khalifeh JM (2020). Reformulation of Degasperis-Procesi field by functional derivatives. Jordan Journal of Physics, 13(1): 67–72. https://doi.org/10.47011/13.1.6

Al-Oqali AD, Al-khamiseh BM, Jaradat EK, and Hijjawi RS (2016). The linear Sigma model Lagrangian density: Fractional formulation. Canadian Journal of Pure and Applied Sciences, 10(1): 3803–3807.

Asjad MI, Faridi WA, Jhangeer A, Abu-Zinadah H, and Ahmad H (2021). The fractional comparative study of the non-linear directional couplers in non-linear optics. Results in Physics, 27: 104459. https://doi.org/10.1016/j.rinp.2021.104459

Bertin MC, Pimentel BM, Valcárcel CE, and Zambrano GER (2017). Hamilton-Jacobi formalism for Podolsky's electromagnetic theory on the null-plane. Journal of Mathematical Physics, 58(8): 082902. https://doi.org/10.1063/1.4999846

Bogdan K and Byczkowski T (2000). Potential theory of Schrodinger operator based on fractional Laplacian. Probability and Mathematical Statistics, 20(2): 293–335.

Cattani C, Martin E, Li M, Liao Z, and Hooshmandasl M (2014). Scaling, self-similarity, and systems of fractional order. Abstract and Applied Analysis, 2014: 843018. https://doi.org/10.1155/2014/843018

Chen D, Sun H, and Zhang Y (2013). Fractional dispersion equation for sediment suspension. Journal of Hydrology, 491: 13–22. https://doi.org/10.1016/j.jhydrol.2013.03.031

Dai J (2021). Stability and Hamiltonian BRST-invariant deformations in Podolsky's generalized electrodynamics. Nuclear Physics B, 971: 115497. https://doi.org/10.1016/j.nuclphysb.2021.115497

Diab AA, Hijjawi RS, Asad JH, and Khalifeh JM (2013). Hamiltonian formulation of classical fields with fractional derivatives: Revisited. Meccanica, 48(2): 323–330. https://doi.org/10.1007/s11012-012-9603-9

Gutiérrez-Vega JC (2007a). Fractionalization of optical beams: I. Planar analysis. Optics Letters, 32(11): 1521–1523. https://doi.org/10.1364/OL.32.001521 **PMid:17546175**

Gutiérrez-Vega JC (2007b). Fractionalization) of optical beams: II. Elegant Laguerre-Gaussian modes. Optics Express, 15(10): 6300–6313. https://doi.org/10.1364/OE.15.006300 **PMid:19546934**

He JH (2014). A tutorial review on fractal space-time and fractional calculus. International Journal of Theoretical Physics, 53(11): 3698-3718. https://doi.org/10.1007/s10773-014-2123-8

Herzallah MAE and Baleanu D (2014). On fractional order hybrid differential equations. Abstract and Applied Analysis, 2014: 389386. https://doi.org/10.1155/2014/389386

Herzallah MAE, Muslih SI, Baleanu D, and Rabei EM (2011). Hamilton–Jacobi and fractional like action with time scaling. Nonlinear Dynamics, 66(4): 549–555. https://doi.org/10.1007/s11071-010-9933-x

Jaradat EK (2017). Hamiltonian formulation for single fluid flows with Caputo's definition. Science International, 29(3): 541-543.

Kruglov SI (2010). Solutions of Podolsky's electrodynamics equation in the first-order formalism. Journal of Physics A: Mathematical and Theoretical, 43: 245403. https://doi.org/10.1088/1751-8113/43/24/245403

La Nave G, Limtragool K, and Phillips PW (2019). Colloquium: Fractional electromagnetism in quantum matter and high-energy physics. Reviews of Modern Physics, 91(2): 021003. https://doi.org/10.1103/RevModPhys.91.021003

Lazar M (2019). Green functions and propagation in the Bopp–Podolsky electrodynamics. Wave Motion, 91: 102388. https://doi.org/10.1016/j.wavemoti.2019.102388

Lazar M and Leck J (2020). Second gradient electromagnet statics: Electric point charge, electrostatic and magneto static dipoles. Symmetry, 12(7): 1104. https://doi.org/10.3390/sym12071104

Magomedov RA, Meilanov RR, Meilanov RP, Akhmedov EN, Beybalaev VD, and Aliverdiev AA (2018). Generalization of thermodynamics in of fractional-order derivatives and calculation of heat-transfer properties of noble gases. Journal of Thermal Analysis and Calorimetry, 133(2): 1189–1194. https://doi.org/10.1007/s10973-018-7024-2

Muslih SI and Baleanu D (2005). Formulation of Hamiltonian equations for fractional variational problems. Czechoslovak Journal of Physics, 55(6): 633–642. https://doi.org/10.1007/s10582-005-0067-1

Novikov VV and Voitsekhovskii KV (2000). Viscoelastic properties of fractal media. Journal of Applied Mechanics and Technical Physics, 41(1): 149–158. https://doi.org/10.1007/BF02465249

Podolsky B and Schwed P (1948). Review of a generalized electrodynamics. Reviews of Modern Physics, 20(1): 40-50. https://doi.org/10.1103/RevModPhys.20.40

Yu R and Wang T (2017). Bertrand's theorem and virial theorem in fractional classical mechanics. The European Physical Journal Plus, 132: 386. https://doi.org/10.1140/epjp/i2017-11639-9