International Journal of Advanced and Applied Sciences, 9(12) 2022, Pages: 57-67

Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

Lindley procedure and MCMC technique in Bayesian estimation for Kumaraswamy Weibull distribution

Fathy H. Eissa [1, 2, *]

[1]Department of Mathematics, College of Science and Arts-Rabigh, King Abdulaziz University, Jeddah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Damanhur University, Damanhur, Egypt

ARTICLE INFO

Article history:
Received 22 January 2022
Received in revised form
15 April 2022
Accepted 20 August 2022

Keywords:
Kumaraswamy Weibull distribution
Type II censored data
Reliability characteristics
Lindley procedure
MCMC technique

ABSTRACT

In this study, a comparison between three methods for estimating unknown parameters of the Kumaraswamy Weibull distribution for different sample sizes of type II censoring data is presented. Specifically, we compare the behaviors of maximum likelihood estimates, Lindley and Markov chain Monte Carlo (MCMC) estimates as Bayesian estimates. We have not found any work on this topic after reviews of the literature except one with little information about the inference of this important distribution. The simplest form for Lindley approximation of the posterior mean is proposed and approximate closed forms of acceptable Bayes estimates for the models of multi-parameters such as Kumaraswamy Weibull distribution is derived. A Monte Carlo simulation is conducted to investigate the performances of the proposed estimators. Finally, three real data examples are analyzed to illustrate the application possibility of the different proposed estimation methods. The results reveal that, although, good performance of the approximate forms of Lindley estimators, the estimators resulting in the MCMC technique are better in the sense of the mean squared errors.

1. Introduction

Kumaraswamy Weibull (KW) distribution is an important extension of the well-known Weibull distribution (Cordeiro et al., 2010; Mitnik, 2013). The cumulative distribution function of KW distribution is,

$$F(x) = 1 - \left[1 - \left(1 - e^{-x^{\gamma}}\right)^{\alpha}\right]^{\beta}, x > 0;\ \gamma, \alpha, \beta > 0 \qquad (1)$$

As a family of distributions, The KW family includes some important well-known distributions such as exponentiated Weibull, exponentiated exponential, exponentiated Rayleigh, Kumaraswamy exponential, Kumaraswamy Rayleigh, Weibull, Rayleigh, and exponential distributions. The distribution function of the Kumaraswamy Weibull distribution given in Eq. 1 has three shape parameters γ, α, and β. For the case of a random variable X following this distribution with the distribution function in Eq. 1, we denote $X \sim KW(\alpha, \beta, \gamma)$. The probability density function (pdf) of the KW distribution is,

$$f(x) = \alpha\beta\gamma x^{\gamma-1} e^{-x^{\gamma}} (1 - e^{-x^{\gamma}})^{\alpha-1} \left[1 - \left(1 - e^{-x^{\gamma}}\right)^{\alpha}\right]^{\beta-1} \qquad (2)$$

for $x > 0$; γ, α and $\beta > 0$. The reliability and failure rate functions are, respectively, given by,

$$R(x) = \left[1 - \left(1 - e^{-x^{\gamma}}\right)^{\alpha}\right]^{\beta} \qquad (3)$$

and

$$h(x) = \alpha\beta\gamma x^{\gamma-1} e^{-x^{\gamma}} \left(1 - e^{-x^{\gamma}}\right)^{\alpha-1} \left[1 - \left(1 - e^{-x^{\gamma}}\right)^{\alpha}\right]^{-1} \qquad (4)$$

for $x > 0$.

It may be useful to mention that the pdf of the KW distribution has a unimodal curve with mode $x_d = [\frac{2(\gamma\alpha-1)}{\gamma\beta(\alpha+1)}]^{\frac{1}{\gamma}}, \gamma\alpha > 1$. The failure rate function $h(x)$ given in Eq. 4 has the following parametric characterizations: (i) it is constant (=β) when $\gamma = \alpha = 1$, (ii) it is increasing (decreasing) when $\gamma \geq 1$ ($\gamma \leq 1$) and $\gamma\alpha \geq 1$ ($\gamma\alpha \leq 1$), (iii) it has bathtub-shape curve when $\gamma > 1$ and $\gamma\alpha < 1$ and (iv) its curve is unimodal when $\gamma < 1$ and $\gamma\alpha > 1$; For details see Eissa (2017).

* Corresponding Author.
Email Address: fkhamis@kau.edu.sa
https://doi.org/10.21833/ijaas.2022.12.008
Corresponding author's ORCID profile:
https://orcid.org/0000-0002-3139-8459

Therefore, the distribution adopts all shapes of failure rates classified into monotone and nonmonotone failure rates, and thence, the KW attracts a wide range of applications in reliability experiments and life testing studies. Practically, it can be useful for modeling and analyzing the life data of many fields such as engineering, medicine, biological sciences, and others. There are few studies about the capability and flexibility for estimating the KW distribution. Cordeiro et al. (2010) showed the availability and flexibility of the estimation using the KW model by the maximum likelihood (ML) method. The Bayes estimates for the parameters of the distribution using the Gibbs sampling procedure are obtained by Mandouh (2016). Mandouh (2016) adopted a non-informative prior that gives little information about the parameters, a priori, and inexact hyperparameters. The main aims of the work are: (1) To propose the simplest form for Lindley approximation of the posterior mean, (2) To examine the performances of the approximate Lindley estimators in the case of the multi-parameter model such as Kumaraswamy Weibull; and (3) To show the capability and flexibility of using the KW distribution for estimation using the well-known Bayesian methods (Huber and Train, 2001) under two loss functions. This may be attracting the attention of the researchers toward the importance of the distribution in analyzing real data.

The contribution of this paper is to provide the simplest form for Lindley approximation of the posterior mean and obtain approximate forms for Bayes estimates using the Lindley procedure in the case of a multi-parameter model such as Kumaraswamy Weibull and compared their performances with the estimates resulting in Markov chain Monte Carlo (MCMC) technique as well as the maximum likelihood estimates. We adopt the quadratic loss function as the asymmetric loss function and the linear exponential (LINEX) loss function as an asymmetric loss function. The Bayes estimate under quadratic loss function for a parameter λ, say, denoted by $\hat{\lambda}_{BS}$ is the posterior mean. The LINEX loss function introduced by Varian (1975) for estimating λ, can be expressed as:

$$L(\Delta) \propto e^{s\Delta} - s\Delta - 1, s \neq 0, \Delta = \hat{\lambda} - \lambda \quad (5)$$

The Bayes estimator for λ, denoted by $\hat{\lambda}_{BL}$, under LINEX loss function is,

$$\hat{\lambda}_{BL} = -\frac{1}{s} \ln \mathrm{E}_{\lambda}(e^{-s\lambda}) \quad (6)$$

assuming that the expected value of $e^{-s\lambda}$ exists and is finite.

The remainder of this paper is organized as follows. In section 2, the maximum likelihood estimates are obtained. The Bayes estimates under squared error and LINEX loss functions are derived in Section 3. Bayesian estimators for the reliability and hazard rate functions are discussed in Section 4. Section 5, provides the conditional distributions required for implementing the Markov chain Monte Carlo to derive the Bayes estimates with respect to squared error and LINEX loss functions. In Section 6, a simulation study is conducted to compare the proposed procedures. In Section 7, three real-life data sets are used to illustrate the application of the proposed inference procedures. At last, the study is concluded in Section 8.

2. Maximum likelihood estimation

Suppose n items are placed on a typical life test. The test is terminated at a preassigned number r of failed items, i.e., the test is terminated at the failure x_r to get a type-II censored sample, $\underline{x} = (x_1, x_2, \dots, x_r)$, of ordered lifetimes, and the remaining (n-r) items are regarded as censored data. Assume the sample $\underline{x}$ from $KW(\alpha, \beta, \gamma)$ distribution with distribution and density functions given, respectively, by Eqs. 1 and 2 and let $\theta = (\alpha, \beta, \gamma)$, then the likelihood function, $L(\theta|\underline{x})$, in this situation, can be given by,

$$L(\theta|\underline{x}) \propto \alpha^r \beta^r \gamma^r e^{-\varphi(\theta)} \quad (7)$$

where,

$$\varphi(\theta) = \sum_{i=1}^{r} x_i^{\gamma} - (\gamma - 1)\sum_1^r \ln x_i - (\alpha - 1)\sum_1^r \ln u_i - (\beta - 1)\sum_1^r \ln(1 - u_i^{\alpha}) - (n - r)\beta \ln(1 - v^{\alpha}),$$
$$u_i = u_i(x_i, \gamma) = 1 - e^{-x_i^{\gamma}} \text{ and } v = v(x_r, \gamma) = 1 - e^{-x_r^{\gamma}}.$$

The log-likelihood function, $l(\theta|\underline{x})$, is,

$$l(\theta|\underline{x}) \propto r \ln \alpha + r \ln \beta + r \ln \gamma - \varphi(\theta). \quad (8)$$

By setting the derivatives of the log-likelihood function with respect to α, β, and γ to zero, the maximum likelihood estimates (MLEs) of the model parameters $\hat{\alpha}_M$, $\hat{\beta}_M$ and $\hat{\gamma}_M$ can be obtained by solving the following non-linear likelihood equations,

$$\frac{r}{\alpha} - (\beta - 1)X_1 + X_2 - (n - r)\beta q_r \ln v = 0,$$
$$\frac{r}{\beta} + X_3 + (n - r) \ln(1 - v^{\alpha}) = 0,$$
$$\frac{r}{\gamma} + X_4 + (\alpha - 1)X_5 - \alpha(\beta - 1)X_6 - (n - r)\alpha\beta p_r q_r = 0 \quad (9)$$

where,

$$X_1 = \sum_1^r q_i \ln u_i, X_2 = \sum_1^r \ln u_i, X_3 = \sum_1^r \ln(1 - u_i^{\alpha}),$$
$$X_4 = \sum_1^r z_i, X_5 = \sum_1^r p_i, X_6 = \sum_1^r p_i q_i,$$
$$p_i = p_i(x_i, \gamma) = x_i^{\gamma} e^{-x_i^{\gamma}} u_i^{-1} \ln x_i, \qquad q_i = q_i(x_i, \alpha, \gamma) = u_i^{\alpha}(1 - u_i^{\alpha})^{-1},$$
$$p_r = p_r(x_r, \gamma) = x_r^{\gamma} e^{-x_r^{\gamma}} v^{-1} \ln x_r, \qquad q_r = q_r(x_r, \alpha, \gamma) = v^{\alpha}(1 - v^{\alpha})^{-1},$$

and

$$z_i = z_i(x_i, \gamma) = (1 - x_i^{\gamma}) \ln x_i.$$

Since there is no closed form of the solution to the above equations, the Newton-Raphson method is widely used to obtain the desired MLEs in such situations. The algorithm of this method can be described as follows:

1. Use the method of moments or any other methods to estimate the parameters α, β, and γ as starting point of iteration, denoted the estimated as $(\alpha_0, \beta_0, \gamma_0)$ and set k=0.
2. Calculate $\left(\frac{\partial l(\theta)}{\partial \alpha}, \frac{\partial l(\theta)}{\partial \beta}, \frac{\partial l(\theta)}{\partial \gamma}\right)_{\alpha_k, \beta_k, \gamma_k}$ and the asymptotic variance-covariance matrix $I^{-1}(\alpha, \beta, \gamma)$. From the well-known observed Fisher Information matrix, the asymptotic variance-covariance matrix for the MLEs is obtained as,

$$\hat{I}^{-1}(\alpha,\beta,\gamma) = \begin{pmatrix} \widehat{var(\alpha)} & \widehat{cov(\alpha,\beta)} & \widehat{cov(\alpha,\gamma)} \\ \widehat{cov(\alpha,\beta)} & \widehat{var(\beta)} & \widehat{cov(\beta,\gamma)} \\ \widehat{cov(\alpha,\gamma)} & \widehat{cov(\beta,\gamma)} & \widehat{var(\gamma)} \end{pmatrix}_{\downarrow(\alpha=\hat{\alpha},\beta=\hat{\beta},\gamma=\hat{\gamma})}$$

3. Update (α, β, γ) as,

$$(\alpha_{k+1}, \beta_{k+1}, \gamma_{k+1}) = (\alpha_k, \beta_k, \gamma_k) + \left(\frac{\partial l(\theta)}{\partial \alpha}, \frac{\partial l(\theta)}{\partial \beta}, \frac{\partial l(\theta)}{\partial \gamma}\right)_{\alpha_k, \beta_k, \gamma_k} \times \hat{I}^{-1}(\alpha,\beta,\gamma).$$

4. Set $k = k + 1$ and then return to the first step.
5. Continue the iterative steps until $|(\alpha_{k+1}, \beta_{k+1}, \gamma_{k+1}) - (\alpha_k, \beta_k, \gamma_k)|$ is smaller than a threshold value. The final estimates of (α, β, γ) are the MLE of the parameters, denoted as $\hat{\alpha}_M$, $\hat{\beta}_M$ and $\hat{\gamma}_M$.

Once MLEs of α, β and γ are obtained, the MLEs of $R(t)$ and $h(t)$ for given mission time t denoted as $\hat{R}_M$and $\hat{h}_M$ can be obtained by the invariance property of MLEs, i.e. replacing α, β and γ by $\hat{\alpha}_M$, $\hat{\beta}_M$ and $\hat{\gamma}_M$ in Eqs. 3 and 4, respectively.

3. Bayesian estimation

Bayesian estimates (Huber and Train, 2001) are quite different from MLE because it takes into consideration both the information from observed sample data and the prior information. Conditional prior distributions were suggested for both α and β given γ, which may appropriately be the gamma distributions with density functions given by,

$$g_1(\alpha|\gamma) = \frac{\alpha^{a-1}}{\Gamma(a)\gamma^a} e^{-\alpha/\gamma}, \alpha > 0, \quad (10)$$

$$g_2(\beta|\gamma) = \frac{\beta^{b-1}}{\Gamma(b)\gamma^b} e^{-\beta/\gamma}, \beta > 0, \quad (11)$$

and the knowledge about γ may be expressed by an exponential distribution with density function,

$$g_3(\gamma) = \frac{1}{c} e^{-\gamma/c}, > 0, \quad (12)$$

where, a, b, and c are hyperparameters of positive values. Adopting the above informative prior distribution for γ with density given by Eq. 12 assumes that it is more convenient to construct prior beliefs about the shape parameter γ, firstly, and conjugate priors for α and β are proposed. If the dependence implicit in Eqs. 10 and 11 is acceptable, the analyst should try to match the experimenter's priors with a selected distribution of the forms in Eqs. 10 and 11. Chosen values of a, b, and c control the precision of the analyst decision. A joint prior density function is then given by

$$g(\alpha,\beta,\gamma) \propto \alpha^{a-1}\beta^{b-1}\gamma^{-(a+b)}e^{-(c\alpha+c\beta+\gamma^2)/c\gamma}, \alpha,\beta,\gamma > 0. \quad (13)$$

Applying Bayes' theorem, the joint posterior density function is given by,

$$p(\alpha,\beta,\gamma|\underline{x}) \propto \alpha^{r+a-1}\beta^{r+b-1}\gamma^{r-a-b}e^{-\varphi-(c\alpha+c\beta+\gamma^2)/c\gamma}, \alpha,\beta,\gamma > 0,$$

where, $\varphi = \varphi(\theta)$ is given in Eq. 7.

3.1. Lindley procedure under squared error loss function

The Bayes estimate, $\hat{w}_B$, of a function $w = w(\alpha, \beta, \gamma)$ under squared error loss function is,

$$\hat{w}_B = \iiint w(\alpha,\beta,\gamma)\ p(\alpha,\beta,\gamma|x)\ d\alpha\ d\beta\ d\gamma \ / \iiint p(\alpha,\beta,\gamma|x)\ d\alpha\ d\beta\ d\gamma.$$

Due to the difficulty of obtaining the integrals contained in $\hat{w}_B$ in closed form, analytically, this study has resorted to use the Lindley procedure (as an approximation method) to approximate the ratio of integrals so that the required estimates can be obtained in usable closed form. Lindley (1980) introduced this procedure to evaluate the forms such as that of the posterior mean of a function $w(\theta|\underline{x})$ and takes the form,

$$E(w(\theta)|\underline{x})) = \int w(\theta)\, e^{q(\theta)} d\theta \ / \int e^{q(\theta)} d\theta, \quad (14)$$

$q(\theta) = l(\theta) + \rho(\theta)$, $l(\theta)$ is the logarithm of the likelihood function and $\rho(\theta)$ is the logarithm of the prior density of θ where $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$. Applying Lindley approximation form expanding about the ML estimate for θ, the posterior mean, $E(w(\theta)|\underline{x}))$, is evaluated by using the form,

$$E(w(\theta)|\underline{x})) = \left[w + (1/2)\textstyle\sum_{i,j}(w_{ij} + 2w_i\rho_i)\sigma_{ij} + (1/2)\sum_{i,j,k,l}(l_{ijk}\sigma_{ij}\sigma_{kl}w_l)\right]_{\theta=\hat{\theta}} + terms\ of\ order\ n^{-2} or\ smaller \quad (15)$$

where, $w = w(\theta)$, $i,j,k,l = 1,2,3,\ldots,n$, $w_i = \partial w/\partial\theta_i$, $w_{ij} = \partial^2 w/\partial\theta_i\,\partial\theta_j$, $l_{ijk} = \partial^3 l(\theta)/\partial\theta_i\,\partial\theta_j\partial\theta_k$, $\rho_j = \partial\rho/\partial\theta_j$, and σ_{ij} is the (i,j)th element in the inverse of the matrix $[-l_{ij}]$. All these are evaluated at the ML estimate of θ, $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n)$. The summation symbols in the form of Equation (15) are abbreviated to one of two or four indexes. The posterior mean $E(w(\theta)|\underline{x})) = \hat{w}_{BS}$, can be rewritten as follows to get the Bayes estimate for $w(\theta)$, under the squared error loss function:

$$\hat{w}_{BS} = w(\theta) + \Phi + \Psi$$

where,

$\Phi = \sum_{i=1,j=1, i\neq j}^{n} w_{ij}\ \sigma_{ij} + (1/2)\sum_{i=1,j=1, i=j}^{n} w_{ij}\ \sigma_{ij}$,
$\Psi = \sum_{i=1}^{n} w_i\ \Psi_i\ , \Psi_i = \sum_{j=1}^{n} \sigma_{ij}\ \mu_j\ , i = 1, 2, 3, \ldots, n$,
$\mu_i = \rho_i + 0.5\ A_i\ , i = 1, 2, 3, \ldots, n$,
$A_k = \sum_{i=1,j=1, i=j}^{n} \sigma_{ij}\ l_{ijk} + 2\sum_{i=1,j=1, i\neq j}^{n} \sigma_{ij}\ l_{ijk}, k = 1, 2, 3, \ldots, n$,

for the symmetric variance-covariance matrix $[-l_{ij}]^{-1}$.

For the case of three parameters, the posterior mean $\hat{w}_{BS}$, the Bayes estimate for $w(\theta)$, under the squared error loss function can be given by,

$$\hat{w}_{BS} = w(\theta) + \Phi + w_1\ \Psi_1 + w_2\ \Psi_2 + w_3\ \Psi_3 \tag{16}$$

where,

$\Phi = w_{12}\sigma_{12} + w_{13}\sigma_{13} + w_{23}\sigma_{23} + (1/2)(w_{11}\sigma_{11} + w_{22}\sigma_{22} + w_{33}\sigma_{33})$,
$\Psi_1 = \sigma_{11}\mu_1 + \sigma_{12}\mu_2 + \sigma_{13}\mu_3$,
$\Psi_2 = \sigma_{21}\mu_1 + \sigma_{22}\mu_2 + \sigma_{23}\mu_3$,
$\Psi_3 = \sigma_{31}\mu_1 + \sigma_{32}\mu_2 + \sigma_{33}\mu_3$,
$\mu_1 = \rho_1 + (0.5)A_1, \mu_2 = \rho_2 + (0.5)A_2, \mu_3 = \rho_3 + (0.5)A_3$,

and,

$A_k = \sum_{i,j=1,i=j}^{3} \sigma_{ij}\ l_{ijk} + 2\sum_{i,j=1,i\neq j}^{3} \sigma_{ij}\ l_{ijk}\ , k = 1,2,3.$

Now the terms of the form of $\hat{w}_{BS}$ are obtained to get the estimates of the parameters α, β, and γ under the squared error loss function as follows.

From the joint prior density function given in Eq. 13, we get,

$\rho_1 = (a-1)\alpha^{-1} - \gamma^{-1},\ \rho_2 = (b-1)\beta^{-1} - \gamma^{-1}$

and,

$\rho_3 = (\alpha + \beta)\gamma^{-2} - (a+b)\gamma^{-1} - c^{-1}$.

The elements σ_{ij}, $i, j = 1,2,3$, are evaluated in terms of the quantities l_{ij} given as follows:

$l_{11} = \frac{-r}{\alpha^2} + (\beta - 1)c_1 + (n-r)\beta c_2, l_{22} = \frac{-r}{\beta^2}$,
$l_{33} = \frac{-r}{\gamma^2} - \sum_1^r y_i + (\alpha - 1)g_1 + \alpha(\beta - 1)m_1 + (n - r)\alpha\beta m_2$,
$l_{12} = l_{21} = -\sum_1^r q_i\ lnu_i - (n-r)\ q_r \ln v$,
$l_{13} = l_{31} = \sum_1^r p_i + (\beta - 1)f_1 + (n-r)\ \beta\ f_2$,
$l_{23} = l_{32} = -\alpha \sum_1^r p_i\, q_i - (n-r)\ \alpha\ p_r q_r$

and the values of l_{ijk} for $i, j, k = 1,2,3$ can be derived as follows:

$l_{111} = \frac{2r}{\alpha^3} - (\beta - 1)\xi_1 - (n-r)\beta\xi_2, l_{222} = \frac{2r}{\beta^3}$,
$l_{333} = \frac{2r}{\gamma^3} - \sum_1^r y_i\ lnx_i + (\alpha - 1)\zeta - \alpha(\beta - 1)\eta_1 - \alpha^2(\beta - 1)\eta_2 - (n-r)\alpha\beta\eta_3 - (n-r)\alpha^2\beta\eta_4$,
$l_{121} = l_{112} = c_1 + (n-r)c_2$,
$l_{131} = l_{113} = (\beta - 1)d_1 + \alpha(\beta - 1)\ d2 + (n-r)\ \beta\ d_3 + (n-r)\alpha\ \beta\ d_4$,
$l_{231} = l_{132} = l_{123} = f_1 + (n-r)f_2$, $l_{331} = l_{133} = g_1 + (\beta - 1)g_2 + (n-r)\beta g_3$,
$l_{332} = l_{233} = \alpha\ m_1 + (n-r)\ \alpha\ m_2$, $l_{221} = l_{122} = l_{232} = l_{223} = 0$,

where,

$\xi_1 = \xi_1(x_i) = \sum_1^r q_i\ (2q_i^2 + 3q_i + 1)(lnu_i)^3, \xi_2 = \xi_1(x_r)$,
$\zeta = \sum_1^r p_i\left(2p_i^2 - 3\ p_i\ z_i + \tau_i\right)(lnx_i)^3$,
$\eta_1 = \eta_1(x_i) = \sum_1^r p_i\ q_i[(\alpha - 1)(\alpha - 2)\ p_i^2 + 3(\alpha - 1)\ z_i\ p_i + \tau_i + 1](lnx_i)^3$,
$\eta_2 = \eta_2(x_i) = \sum_1^r {p_i}^2\ {q_i}^2[2\alpha\ p_i\ q_i + 3(\alpha - 1)\ p_i + 3z_i](lnx_i)^3$,
$\eta_3 = \eta_1(x_r), \eta_4 = \eta_2(x_r)$,
$c_1 = c_1(x_i) = -\sum_1^r q_i\ (q_i + 1)(lnu_i)^2, c_2 = c_1(x_r)$,
$d_1 = d_1(x_i) = -2\sum_1^r p_i\ q_i\ (q_i + 1)\ lnu_i\ lnx_i$,
$d_2 = d_2(x_i) = -\sum_1^r p_i\ q_i\ (2{q_i}^2 + 3q_i + 1)(lnu_i)^2\ lnx_i$,
$d_3 = d_1(x_r), d_4 = d_2(x_r)$,
$f_1 = f_1(x_i) = -\sum_1^r p_i\ q_i\ [\alpha\ (q_i + 1)\ lnu_i + 1]\ lnx_i$, $f_2 = f_1(x_r)$
$g_1 = \sum_1^r p_i\ \ (z_i - p_i)(lnx_i)^2$,
$g_2 = g_2(x_i) = -\sum_1^r p_i\ q_i\ [2\alpha\ p_i\ q_i + (2\alpha - 1)\ p_i + z_i](lnx_i)^2$
$g_3 = g_3(x_i) = -\sum_1^r p_i\ q_i\ [(2\alpha\ p_i\ q_i + (\alpha - 1)p_i + z_i)(q_i + 1)]lnu_i(lnx_i)^2$
$g_4 = g_2(x_r), g_5 = g_3(x_r)$
$m_1 = m_1(x_i) = -\sum_1^r p_i\ q_i\ [\alpha\ p_i\ q_i + (\alpha - 1)p_i + z_i](lnx_i)^2$,
$m_2 = m_1(x_r)$,
$y_i = x_i^\gamma\,(lnx_i)^2, z_i = \left(1 - x_i^\gamma\right), \tau_i = (x_i^{2\gamma} - 3x_i^\gamma + 1)$,
$p_i = p_i(x_i) = x_i^\gamma e^{-x_i^\gamma} u_i^{-1}$, and $q_i = q_i(x_i) = u_i^\alpha(1 - u_i^\alpha)^{-1}$.

Therefore, the Bayes estimates can be obtained for the parameters α, β, and γ by substituting the required values cited above in Eq. 16, in what follows:

-If $w(\theta) = \alpha$, then $\hat{\alpha}_{BS} = \alpha + \Psi_1$. (17)
-If $w(\theta) = \beta$, then $\hat{\beta}_{BS} = \beta + \Psi_2$. (18)
-If $w(\theta) = \gamma$, then $\hat{\gamma}_{BS} = \gamma + \Psi_3$. (19)

Keeping in mind, these estimates are evaluated at the ML estimates of the parameters.

3.2. Lindley procedure under LINEX loss function

Under the LINEX loss function, the Bayes estimate, $\hat{w}_{BL}$ is given by,

$$\hat{w}_{BL} = (-1/s)\ ln\,E(e^{-sw(\theta)}|\underline{x})\ , s \neq 0 \tag{20}$$

where,

$E(e^{-sw(\theta)}|\underline{x}) = \iiint e^{-sw(\theta)}\ p\left(\alpha, \beta, \gamma|\underline{x}\right)d\alpha d\beta d\gamma / \iiint p\left(\alpha, \beta, \gamma|\underline{x}\right)d\alpha d\beta d\gamma$.

Applying Lindley approximation on this ratio of integration, we can get approximated form of the posterior mean as,

$$E\left(e^{-sw(\theta)}|\underline{x}\right) = e^{-sw(\theta)} + \Phi + w_1\ \Psi_1 + w_2\ \Psi_2 + w_3\ \Psi_3$$

where, Φ, Ψ_1, Ψ_2 and Ψ_3 are defined in Eq. 16.

Therefore, the Bayes estimates for the parameters α, β and γ under the LINEX loss function are in what follows:

- When $w(\theta) = e^{-s\alpha}$, we get:

$$\hat{\alpha}_{BL} = \alpha - s^{-1}\ ln[(0.5)s^2\sigma_{11} - s\ \Psi_1 + 1]\ , s \neq 0. \tag{21}$$

- When $w(\theta) = e^{-s\beta}$, we get:

$$\hat{\beta}_{BL} = \beta - s^{-1}\,ln[(0.5)s^2\sigma_{22} - s\,\Psi_2 + 1]\,, s \neq 0. \tag{22}$$

- When $w(\theta) = e^{-s\gamma}$, we get:

$$\hat{\gamma}_{BL} = \gamma - s^{-1}\,ln[(0.5)s^2\sigma_{33} - s\,\Psi_3 + 1]\,, s \neq 0. \tag{23}$$

Remembering that these estimates are evaluated at $\hat{\alpha}_M, \hat{\beta}_M, \hat{\gamma}_M$.

4. Bayesian estimates for R and h

As parameters, the Bayes estimates for $R = R(x)$ and $h = h(x)$ are derived under the two loss functions in what follows.

4.1. Under squared error loss function

From Eq. 16, we derive the required estimates as follows:

- When $w(\theta) = R$, we get:

$$\hat{R}_{BS} = R(1 + 0.5\,Q_1 + Q_2 + Q_3) \tag{24}$$

where,

$Q_1 = \sigma_{11}\beta q(ln\,u)^2[(\beta-1)q - 1] + \sigma_{22}\beta^{-2}(ln\,R)^2 + \sigma_{33}\alpha\beta pq[\alpha(\beta-1)pq - (\alpha-1)p - z]$,
$Q_2 = \sigma_{13}\beta pq[\alpha(\beta-1)q\,ln\,u - \alpha\,ln\,u - 1] - \sigma_{23}\alpha pq(1 + ln\,R) - \sigma_{12}q\,(1 + ln\,R)\,ln\,u$,
$Q_3 = -\beta q\,(ln\,u)\,\Psi_1 + \beta^{-1}\,(ln\,R)\,\Psi_2 - \alpha\beta pq\Psi_3$,

Ψ_1, Ψ_2 and Ψ_3 are defined in Eq. 16.

- When $w(\theta) = h$, we get:

$$\hat{h}_{BS} = h(1 + 0.5\,K_1 + K_2 + K_3) \tag{25}$$

where,

$K_1 = \sigma_{11}[(2q^2 + 3q + 1)(ln\,u)^2 + 2\alpha^{-1}(q+1)\,ln\,u] + \sigma_{33}[2\alpha^2p^2q^2 + (\alpha-1)\,p^2(3\alpha\,q + \alpha - 2) + 2\gamma^{-1}(t_2 - \gamma^{-1}) + 3(\alpha\,q + \alpha - 1)z\,p + t_3]$,
$K_2 = \sigma_{12}\beta^{-1}t_1 + \sigma_{13}[\gamma^{-1}t_1 + \alpha^{-1}(z - p) + (q+1)(2p - p\,ln\,u + z\,ln\,u) + \alpha p(2q^2 + 3q + 1)\,ln\,u] + \sigma_{23}\beta^{-1}t_2$,
$K_3 = t_1\Psi_1 + \beta^{-1}\Psi_2 + t_2\Psi_3$,
$t_1 = \alpha^{-1} + (q+1)\,ln\,u$, $t_2 = \alpha\,p\,q + (\alpha-1)p + z + \gamma^{-1}$,
and,

$$t_3 = (x_i^{2\gamma} - 3x_i^{\gamma} + 1)(lnx_i)^2.$$

4.2. Under LINEX loss function

From Eq. 20, we derive the required estimates as follows:

- When $w(\theta) = e^{-sR}$, we get:

$$\hat{R}_{BL} = R - s^{-1}\,ln(1 + s\,R\,Q) \tag{26}$$

where,

$Q = 0.5\,s\,Q_4 + s\,Q_5 - 0.5\,Q_1 - Q_2 - Q_3$,
$Q_4 = \sigma_{11}\beta^2q^2\,R\,(ln\,u)^2 + \sigma_{22}\beta^{-2}\,R\,(ln\,R)^2 + \sigma_{33}\alpha^2\,\beta^2\,p^2\,q^2\,R$,
$Q_5 = \sigma_{13}\,\alpha\,\beta^2\,p\,q^2R\,ln\,u - \sigma_{23}\alpha\,p\,qR\,ln\,R - \sigma_{12}q\,ln\,u\,ln\,R$,

Q_1, Q_2 and Q_3 are given in Eq. 24.

- When $w(\theta) = e^{-sh}$, we get:

$$\hat{h}_{BL} = h - s^{-1}\,ln(1 + s\,h\,K) \tag{27}$$

where,

$K = 0.5\,K_4 + K_5 - 0.5\,K_1 - K_2 - K_3$,
$K_4 = s\,h(\sigma_{11}\,{t_1}^2 + \sigma_{22}\,\beta^{-2} - \sigma_{33}{t_2}^2)$,
$K_5 = \sigma_{12}\,\beta^{-1}\,t_1 + \sigma_{13}\,t_1\,t_2 + \sigma_{23}\,\beta^{-1}\,t_2$.

K_1, K_2, K_3, t_1 and t_2 are given by in Eq. 25.

All quantities including Eqs. 24-27 were evaluated at the ML estimates of the parameters α, β, and γ. The estimators of R and h are evaluated at a certain point in time $x = x_0$. That is, $p = p(x_0, \hat{\gamma}_M)$, $q = q(x_0, \hat{\alpha}_M, \hat{\gamma}_M)$, and $z = z(x_0, \hat{\gamma}_M)$ given as in Eq. 9.

5. MCMC technique

MCMC is a popular and useful method because it has been widely used in statistics for estimating complex Bayesian problems. In this section, the MCMC method namely Metropolis-Hastings within Gibbs sampling algorithm (Tierney, 1994) is applied to generate posterior samples and then compute the Bayes estimates of α, β, and γ. The conditional posterior distributions of α, β and γ can be obtained from $p(\alpha, \beta, \gamma|x)$ as:

$$p_1(\alpha|\beta,\gamma,\underline{x}) \propto \alpha^{r+a-1}\left[e^{-\frac{\alpha}{\gamma}}\right](1 - v^{\alpha})^{\beta(n-r)}\prod_{i=1}^{r}({u_i}^{\alpha})(1 - {u_i}^{\alpha})^{\beta-1}\,, \tag{28}$$

$$p_2(\beta|\alpha,\gamma,\underline{x}) \propto \beta^{r+b-1}e^{-\beta[\frac{1}{\gamma}-(n-r)\ln(1-v^{\alpha})-(\sum_{i-1}^{r}(1-u_i)^{\alpha})]} \tag{29}$$

and,

$$p_3(\gamma|\alpha,\beta,\underline{x}) \propto \gamma^{r-a-b}[(1 - v^{\alpha})^{\beta(n-r)}]\left[e^{-(\frac{\alpha c+\beta c+\gamma^2}{\gamma c})}\right]\left(\prod_{=1}^{r}u_i^{(\alpha-1)}(1 - u_i^{\alpha})^{\beta-1}\right), \tag{30}$$

where, u_i and v are given in Eq. 7. From Eq. 32, the conditional posterior density of β is $Gamma\left(r + b, \frac{1}{\gamma} - (n-r)\ln(1 - v^{\alpha}) - \sum_{i=1}^{r}(1 - u_i)^{\alpha}\right)$. Thus, the samples of β can be generated by using any Gamma sub-routine. On the other side, From Eqs. 28 and 30, the conditional posterior density of α and γ do not present standard form, but the plots of them show that they are similar to the normal distribution. Thus $p_1(\alpha|\beta,\gamma,\underline{x})$ and $p_3(\gamma|\alpha,\beta,\underline{x})$ are log-concave. i.e., all the conditional posterior distributions contain a single maximum value, and this allows us to apply MCMC.

To simulate random samplesfrom $p_1(\alpha|\beta,\gamma,\underline{x})$ and $p_3(\gamma|\alpha,\beta,\underline{x})$, we use the MH algorithm with the normal proposal distribution $N(\hat{\alpha}_M, Var\hat{\alpha}_M)$ and $N(\hat{\gamma}_M, Var\hat{\gamma}_M)$ through these conditional posterior densities. The algorithm is detailed in the following steps:

(1) Set the initial guess $(\alpha^0, \beta^0, \gamma^0) = (\hat{\alpha}_M, \hat{\beta}_M, \hat{\gamma}_M)$
(2) Set $i = 1$.
(3) Generate $\beta^{(i)}$ from $Gamma\left(r + b, \frac{1}{\gamma} - (n-r)\ln(1-v^{\alpha}) - \sum_{i=1}^{r}(1-u_i)^{\alpha}\right)$.
(4) Using MH:

(a) Generate α^* from $N(\alpha^{(i-1)}, Var(\alpha))$ and γ^* from $N(\gamma^{(i-1)}, Var(\gamma))$.
(b) Evaluate the acceptance probabilities,

$$\Omega_\alpha = min\left[1, \frac{p_1(\alpha^*|\beta^{(i)}, \gamma^{(i-1)}, \underset{\sim}{x})}{p_1(\alpha^{(i-1)}|\beta^{(i)}, \gamma^{(i-1)}, \underset{\sim}{x})}\right],$$
$$\Omega_\gamma = min\left[1, \frac{p_3(\gamma^*|\alpha^{(i)}, \beta^{(i)}, \underset{\sim}{x})}{p_3(\gamma^{(i-1)}|\alpha^{(i)}, \beta^{(i)}, \underset{\sim}{x})}\right].$$

(c) Generate a u_1 and u_2 from a Uniform (0,1).
(d) If $u_1 < \Omega_\alpha$, accept the proposal, and set $\alpha^{(j)} = \alpha^*$, else set $\alpha^{(j)} = \alpha^{(j-1)}$.
(e) If $u_2 < \Omega_\gamma$, accept the proposal, and set $\gamma^{(j)} = \gamma^*$, else set $\gamma^{(j)} = \gamma^{(j-1)}$.

(5) Compute the reliability characteristics $R(t)$ and $h(t)$ as,

$$R^{(i)}(x) = \left[1 - \left(1 - e^{-x^{\gamma^{(i)}}}\right)^{\alpha^{(i)}}\right]^{\beta^{(i)}},$$

and,

$$h^{(i)}(x) = \alpha^{(i)}\beta^{(i)}\gamma^{(i)}x^{\gamma^{(i)}-1}e^{-x^{\gamma^{(i)}}}(1 - e^{-x^{\gamma^{(i)}}})^{\alpha^{(i)}-1}\left[1 - \left(1 - e^{-x^{\gamma^{(i)}}}\right)^{\alpha^{(i)}}\right]^{-1}.$$

(6) Set $i = i + 1$.
(7) Duplicate steps 2-6 N_1 times to obtain:

$(\alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}, R^{(1)}(x), h^{(1)}(x)), \ldots,$
$(\alpha^{(N_1)}, \beta^{(N_1)}, \gamma^{(N_1)}, R^{(N_1)}(x), h^{(N_1)}(x))$.

Then, the Bes of $\Lambda = \alpha, \beta, \gamma, R(x)\ or\ h(x)$ under SE, LINEX loss functions are, respectively, given by,

$$\Lambda_{SE} = \frac{1}{N_1 - N_0}\sum_{i=N_0+1}^{N_1}\Lambda^{(i)}$$

and,

$$\Lambda_{LINEX} = -\frac{1}{s}\ln\left(\frac{1}{N_1 - N_0}\sum_{i=N_0+1}^{N_1}e^{-s\Lambda^{(i)}}\right)$$

where, N_0 is burn-in period.

6. Simulation study

In the above sections, mathematical forms were derived for different estimates of the parameters of the KW distribution and its reliability and failure rate functions as parameters. It is very difficult to compare and assess the performances of these estimates theoretically. Therefore, extensive numerical experiments were conducted by Monte Carlo simulation study to carry out the comparisons and assessments mainly with respect to the mean squared errors of the different estimates based on different sample sizes. Different sample sizes ranging from small to large were generated.

For a given value of c, a value of γ was generated from Eq. 12 and used with given values of a and b to generate values of α and β. The suggested values of the prior parameters are: $a = 3$, $b = 2$ and $c = 3$ which yield $\gamma = 2.1016$, $\alpha = 3.6311$ and $\beta = 0.7283$ (as simulated true values) and the simulated true values of $R(x)$ and $h(x)$ are $R(1.5) = 0.4226$ and $h(1.5) = 3.1268$. Monte Carlo simulations are carried out utilizing 1000 type II censored samples for each simulation. We assume that the Bayes estimates are obtained under the squared error (SE) loss function and LINEX loss function (s=25, -25) together. Also, we compute the Bayes estimates based on 10000 MCMC samples and a 2000 burn-in period i.e. N_1=10000 and N_0=2000.

Samples of sizes: 15, 25, 50, and 100 were generated from the $KW(\alpha, \beta, \gamma)$ and employed the MATHEMATICA 10 to perform all computations. By solving the nonlinear Eq. 7, the MLEs $\hat{\alpha}_M$, $\hat{\beta}_M$ and $\hat{\gamma}_M$ were estimated. The estimates $\hat{R}_M(x_0)$ and $\hat{h}_M(x_0)$ are then evaluated at a time point $x = 1.5$. The mean values of the different Bayes estimates were evaluated. The associated mean squared errors (MSEs) of the different estimates were calculated to compare the performances of the resulting estimators. The results are reported in Tables 1-5.

From the simulation results, we can state the following remarks:

1. The results in all tables reveal that the MSEs of the estimators tend to decrease by increasing the sample size and this indicates that the estimators for all considered parameters are consistent.
2. Bayes estimates under the Lindley procedure and MCMC technique have the smallest MSEs for α, β, γ, R(t), and h(t). Hence, Bayes estimates perform better than the MLEs in all cases considered.
3. Bayes estimates using the MCMC technique perform better than Bayes estimates using the Lindley procedure in the sense of having smaller MSEs.
4. Bayes estimates under the LINEX loss function work the best in all cases of sample sizes under consideration and for all the parameters, followed by the estimates under the squared error loss function.
5. Bayes estimates under LINEX loss function with s=25 are provides better estimates in the sense of having smaller MSEs.
6. As usual, the MLEs work well at moderate and large sample sizes for all α, β, γ, R(t), and h(t).

Table 1: Average of different estimates and MSEs (in brackets) for γ

n	r	MLE	Lindley			MCMC		
			SE	LINEX		SE	LINEX	
				$s=-25$	$s=25$		$s=-25$	$s=25$
15	10	1.9241 (0.1247)	2.0451 (0.1045)	1.9782 (0.0996)	1.9920 (0.0904)	1.7998 (0.0895)	1.8969 (0.0815)	2.1337 (0.0746)
25	20	2.0856 (0.1149)	1.945 (0.0935)	2.2012 (0.0899)	2.0147 (0.0796)	1.8973 (0.0805)	2.2198 (0.0758)	1.9999 (0.0683)
50	40	2.0277 (0.0958)	2.0249 (0.0834)	2.1457 (0.0746)	1.9993 (0.0697)	2.2140 (0.0779)	1.9647 (0.0688)	2.0128 (0.0534)
100	80	1.9784 (0.0598)	2.0728 (0.0526)	2.1255 (0.0485)	2.0967 (0.0411)	2.1999 (0.0479)	1.9483 (0.0399)	1.9988 (0.0347)

Table 2: Average of different estimates and MSEs (in brackets) for α

n	r	MLE	Lindley			MCMC		
			SE	LINEX		SE	LINEX	
				$s=-25$	$s=25$		$s=-25$	$s=25$
15	10	4.2451 (0.9467)	4.1558 (0.9146)	3.8942 (0.9100)	3.7214 (0.8266)	4.2789 (0.8012)	3.8217 (0.7541)	4.0012 (0.6901)
25	20	4.2011 (0.8997)	3.8199 (0.8277)	3.8145 (0.7966)	3.5489 (0.7351)	3.9012 (0. 6989)	3.7461 (0.6462)	3.6890 (0.5763)
50	40	3.9862 (0.8532)	4.0085 (0.7933)	3.7985 (0.7463)	3.6999 (0.6654)	3.7233 (0.6187)	3.8720 (0.5871)	3.5577 (0.4987)
100	80	3.7666 (0.6473)	3.5341 (0.6178)	3.6444 (0.5888)	3.7001 (0.5100)	3.5664 (0.4399)	3.7014 (0.4023)	3.6982 (0.3075)

Table 3: Average of different estimates and MSEs (in brackets) for β

n	r	MLE	Lindley			MCMC		
			SE	LINEX		SE	LINEX	
				$s=-25$	$s=25$		$s=-25$	$s=25$
15	10	0.9871 (0.1109)	0.9211 (0.0958)	0.8874 (0.0745)	0.9147 (0.0694)	0.8479 (0.0652)	0.8277 (0.0619)	0.7568 (0.0584)
25	20	0.9233 (0.0987)	0.8366 (0.0742)	0.9472 (0.0621)	0.8914 (0.0591)	0.7844 (0.0513)	0.8013 (0.0492)	0.7782 (0.0442)
50	40	0.9322 (0.0874)	0.7495 (0.0589)	0.7914 (0.0510)	0.7483 (0.0411)	0.7256 (0.0402)	0.7548 (0.0371)	0.6914 (0.0291)
100	80	0.8120 (0.0614)	0.7511 (0.0564)	0.7374 (0.0451)	0.6577 (0.0354)	0.6733 (0.0309)	0.6691 (0.0296)	0.6257 (0.0228)

Table 4: Average of different estimates and MSEs (in brackets) for $R(1.5)$

n	r	MLE	Lindley			MCMC		
			SE	LINEX		SE	LINEX	
				$s=-25$	$s=25$		$s=-25$	$s=25$
15	10	0.4527 (0.0535)	0.4158 (0.0311)	0.3856 (0.0295)	0.3988 (0.0251)	0.4293 (0.0214)	0.4157 (0.0190)	0.3999 (0.0153)
25	20	0.3942 (0.0472)	0.3851 (0.0289)	0.4013 (0.0228)	0.4019 (0.0199)	0.3955 (0.0179)	0.4298 (0.0122)	0.3947 (0.0101)
50	40	0.4155 (0.0359)	0.3912 (0.0214)	0.3794 (0.0198)	0.3879 (0.0138)	0.3985 (0.0111)	0.3977 (0.0086)	0.4028 (0.0071)
100	80	0.3799 (0.0264)	0.4153 (0.0159)	0.4255 (0.0136)	0.3999 (0.0100)	0.4187 (0.0087)	0.4093 (0.0055)	0.4229 (0.0430)

Table 5: Average of different estimates and MSEs (in brackets) for $h(1.5)$

n	r	MLE	Lindley			MCMC		
			SE	LINEX		SE	LINEX	
				$s=-25$	$s=25$		$s=-25$	$s=25$
15	10	3.1451 (1.0124)	3.1478 (0.9254)	3.2199 (0.8954)	3.2781 (0.8147)	3.3412 (0.7985)	3.1479 (0.7533)	3.2945 (0.5471)
25	20	3.1514 (0.9315)	2.9854 (0.7549)	3.3154 (0.7162)	3.1486 (0.6794)	3.1222 (0.6247)	3.2954 (0.5963)	3.1456 (0.5132)
50	40	3.2552 (0.8924)	2.8964 (0.6955)	2.9614 (0.6634)	3.1745 (0.6147)	3.3475 (0.5896)	3.2147 (0.4965)	3.1115 (0.3966)
100	80	3.1555 (0.0764)	3.0417 (0.5584)	3.1254 (0.4987)	2.9999 (0.4388)	3.2990 (0.4102)	3.1845 (0.3821)	3.2247 (0.3145)

7. Applications

This study has used three real data sets as examples to illustrate the availability and flexibility to apply the different estimates proposed in the above sections. Moreover, this study has aimed to illustrate the eligibility of the different estimates for the parameters of each data set. This study has adopted estimations for the parameters using the uncensored and censored schemes for these data sets. With respect to the Bayesian estimates for the parameters of these data sets (reported in Tables 7, 9, and 11 below), non-informative priors were considered for the parameters α, β and γ in order to avoid or reduced the impact of the prior's parameters on the posterior. This study has used non-informative priors with $a = b = c = 0.001$. The sizes of the data sets are between small to large and very large.

Application 1. The data set was considered to correspond to remission times (in months) of a random sample of 128 bladder cancer patients. The data were previously studied by Lee and Wang (2003), Zea et al. (2012), and Najarzadegan et al. (2017).

The data set are as follows: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Some statistical properties of this data set and the values of Kolmogorov-Smirnov (K-S) statistics with their respective p-value are shown in Table 6. The notations used in Table 6: $\overline{x}$, $E(X)$, Var, Sk and Ku are the sample mean, the expected value, variance, skewness, and kurtosis, respectively.

The values of K-S and p-value reveal that the KW distribution provides an excellent fit to this data. Fig. 1 supports this claim. Table 7 reports the MLEs and the Bayesian estimates for the different parameters under censored schemes when n=128 and r=96. The estimates of the survival time R and the risk time h are evaluated at the time point $x_0 = 2$.

Table 6: Remission times of bladder cancer data: Statistical properties

$\overline{x}$	$E(X)$	Var	Sk	Ku	*K-S test value*	*p-value*
9.365	8.672	110.425	3.286	18.483	0.061	0.707

Table 7: Remission times of bladder cancer data: ML and Bayesian estimates

Parameter	MLE	Lindley			MCMC		
		SE	LINEX		SE	LINEX	
			s= -500	s=500		s= -500	s=500
α	4.3679	4.6205	4.5327	4.3467	4.4524	4.3891	4.5983
β	0.4293	0.4162	0.4025	0.4183	0.4255	0.4352	0.4234
γ	0.6138	0.5981	0.6177	0.6043	0.6214	0.6155	0.5894
$R(2)$	0.8341	0.8459	0.8269	0.8215	0.8322	0.8234	0.8201
$h(2)$	0.2558	0.2384	0.2258	0.2134	0.2202	0.2153	0.2057

Application 2. The second data set corresponds to the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed. The data were studied previously by many authors such as Barlow et al. (1984) and Andrews and Herzberg (2012).

The data set are as follows: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Table 8 lists the values of some statistical properties of this data set and the values of the K-S statistic with their respective p-value. The values of K-S and p-value reveal that the KW distribution provides a significantly fit this data. Fig. 1 supports this claim. Table 9 reports the MLEs and the Bayesian estimates for the parameters under censored schemes n=76 and r =57. The estimates of R and h are evaluated at the time point $x_0 = 0.5$.

Table 8: Data of fatigue structure of Kevlar 373/epoxy: Statistical properties

$\overline{x}$	$E(X)$	Var	SK	Ku	*K-S test value*	*p-value*
1.959	1.959	2.477	1.979	8.161	0.098	0.428

Table 9: Data of fatigue structure of Kevlar 373/epoxy: ML and Bayesian estimates

Parameter	MLE	Lindley			MCMC		
		SE	LINEX		SE	LINEX	
			s= -500	s=500		s= -500	s=500
α	4.1217	4.1205	4.2145	4.1056	4.1568	4.2111	4.0173
β	2.4011	2.2780	2.3941	2.3618	2.2879	2.2541	2.1888
γ	0.5825	0.4218	0.4912	0.5746	0.4588	0.5134	0.4697
$R(0.5)$	0.8805	0.8330	0.8549	0.8738	0.8344	0.8321	0.8287
$h(0.5)$	0.2205	0.2526	0.2566	0.2272	0.2489	0.2384	0.2278

Application 3. As a third application, this study has considered the real data set that was studied previously by Gross and Clark (1975). The data represents the relief times of twenty patients receiving an analgesic. The data set is as follows: 1.1,

1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

Fig. 1 supports the claim that the KW distribution provides a significantly fit to this data through the values of the K-S test and its p-value cited in Table 10. Table 11 summarizes the performances of the different estimation methods for this data set when n=20 and r =16. The estimates of the parameters R and h are estimated at $x_0 = 1.7$.

Table 10: Data of relief times of twenty patients: Statistical properties

$\bar{x}$	$E(X)$	Var	SK	Ku	*K-S test value*	*p-value*
1.900	1.923	0.496	1.720	5.924	0.181	0.474

Table 11: Data of relief times of twenty patients: ML and Bayesian estimates

Parameter	MLE	Lindley			MCMC		
		SE	LINEX		SE	LINEX	
			s= -500	s=500		s= -500	s=500
α	7.5221	5.8547	6.6459	6.4943	6.9543	6.8469	6.4865
β	0.3060	0.2279	0.3022	0.2927	0.2874	0.2147	0.2103
γ	2.5780	2.2891	2.2145	2.5608	2.6412	2.4561	2.3412
$R(1.7)$	0.5465	0.5858	0.5681	0.5278	0.5735	0.5547	0.5263
$h(1.7)$	2.9028	2.3836	2.7362	2.9414	2.3188	2.5453	2.4128

Based on the estimated values of the parameters for the above data sets, and in view of the results of Eissa's (2017) Theorem 4, this study shows that data sets 1 and 2 have upside-down bathtub-shaped hazard rates while data set 3 has increasing hazard rate. This result is shown in Fig. 2.

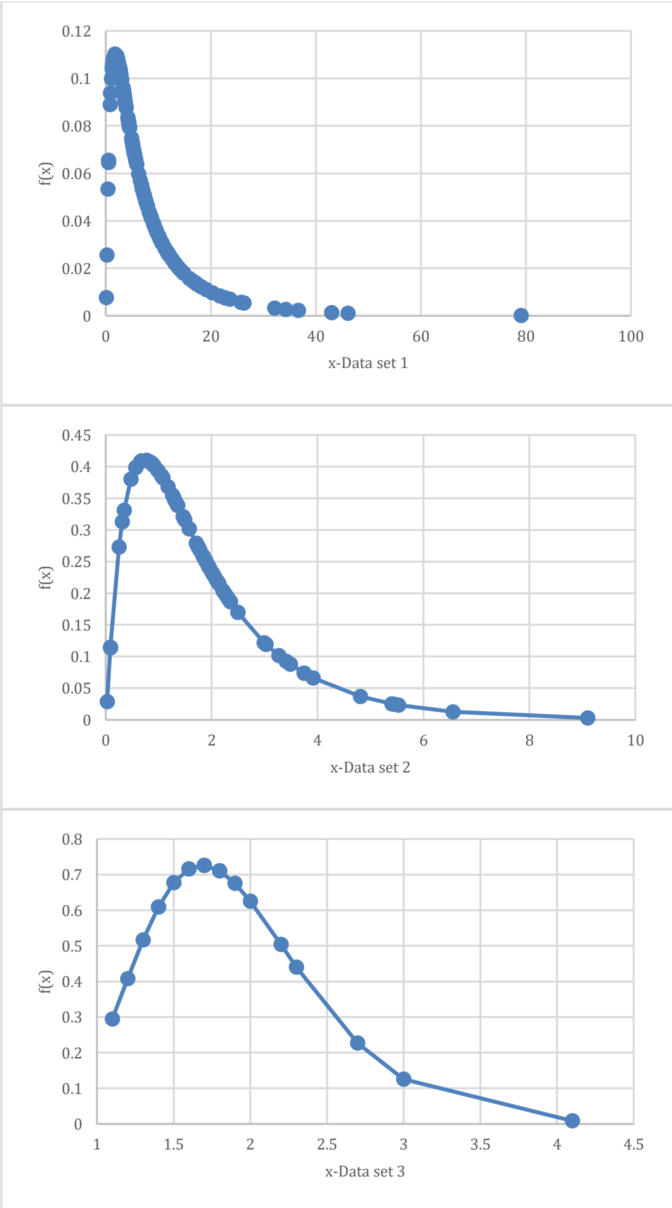

Fig. 1: Pdfs functions for the real data sets

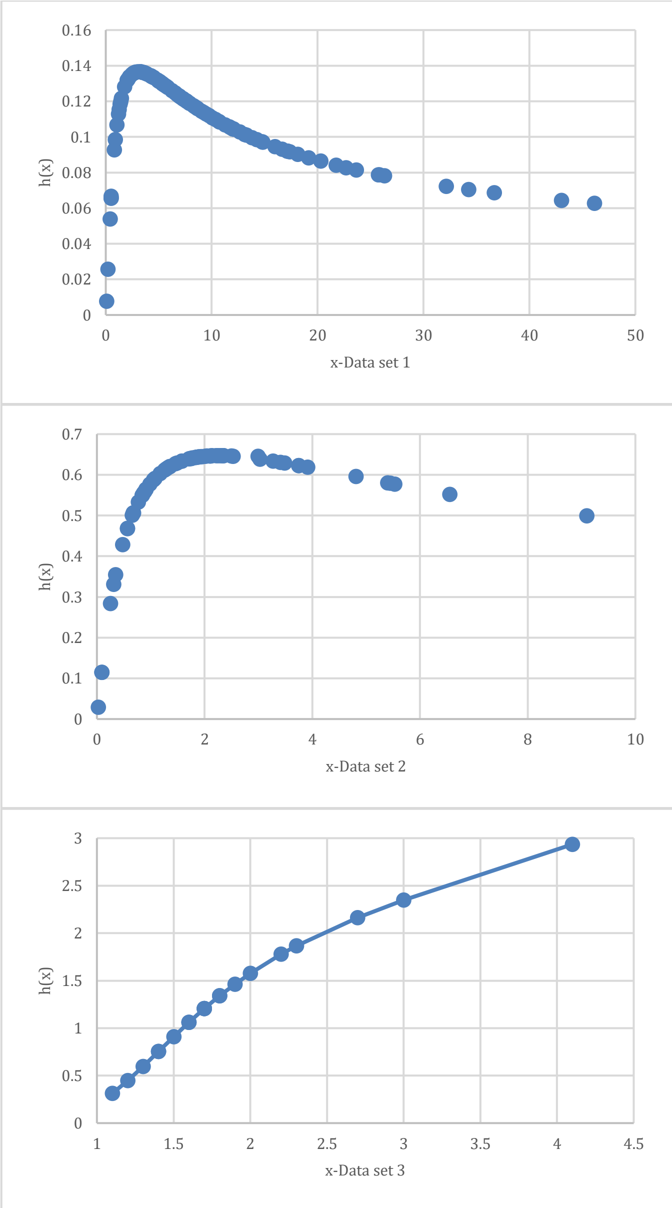

Fig. 2: Hazard rate functions for the real data sets

8. Conclusion

In conclusion, the core emphasis was on the classical and Bayesian estimation (Huber and Train, 2001) for the Kumaraswamy Weibull lifetimes (Cordeiro et al., 2010; Mitnik, 2013) to show the availability and flexibility of the Kumaraswamy Weibull distribution as a model for analyzing lifetime data sets in presence of right censoring. Although the existence of the three shape parameters, the Lindley procedure was adopted to obtain approximate forms for different estimators in simple mathematical formats. The MCMC technique was also adopted to get the estimates directly from the posterior distribution by generating posterior samples. Bayesian estimates for the shape parameters and the reliability and hazard rate functions (as parameters) were derived under the squared error loss and LINEX loss functions (Varian, 1975). The performances of these estimators had been compared with the MLEs for all parameters based on the MSEs. Based on the simulation experiments, the approximate Lindley estimators for different parameters perform better than MLEs for all sample sizes, especially under the LINEX loss function. Although, good performance of the approximate forms of Lindley estimators, the MCMC technique performs better than it in all cases considered.

Analyzing the three real data sets shows the applicability of the proposed estimators. Adopting the approximate forms for different Bayes estimates using the Lindley procedure, seem to be appropriate and adequate (in terms of MSEs) to obtain acceptable Bayes estimates for models of multi-parameters such as the Kumaraswamy Weibull distribution as shown in this study. However, The MCMC technique is the best.

Data availability

The data in application 1 is openly available from the book by Lee and Wang (2003). The data in application 2 is openly available in Barlow et al.

(1984). The data in application 3 is openly available from the book by Gross and Clark (1975).

Acknowledgment

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant no. (G: 528-662-1439). The authors acknowledge thanks to DSR financial support.

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

References

Andrews DF and Herzberg AM (2012). Data: A collection of problems from many fields for the student and research worker. Springer Science and Business Media, Berlin, Germany.

Barlow RE, Toland RH, and Freeman, T (1984). A Bayesian analysis of stress-rupture life of Kevlar 49/epoxy spherical pressure vessels. In the 5th Canadian Conference in Applied Statistics, Marcel Dekker, New York, USA.

Cordeiro GM, Ortega EM, and Nadarajah S (2010). The Kumaraswamy Weibull distribution with application to failure data. Journal of the Franklin Institute, 347(8): 1399-1429. https://doi.org/10.1016/j.jfranklin.2010.06.010

Eissa FH (2017). The exponentiated Kumaraswamy Weibull distribution with application to real data. International Journal of Statistics and Probability, 6: 167-182. https://doi.org/10.5539/ijsp.v6n6p167

Gross AJ and Clark V (1975). Survival distributions: Reliability applications in the biomedical sciences. John Wiley and Sons, Hoboken, USA.

Huber J and Train K (2001). On the similarity of classical and Bayesian estimates of individual mean partworths. Marketing Letters, 12(3): 259-269. https://doi.org/10.1023/A:1011120928698

Lee ET and Wang J (2003). Statistical methods for survival data analysis. Volume 476, John Wiley and Sons, Hoboken, USA. https://doi.org/10.1002/0471458546

Lindley DV (1980). Approximate Bayesian methods. Trabajos de Estadística y de Investigación Operativa, 31(1): 223-245. https://doi.org/10.1007/BF02888353

Mandouh R (2016). Bayesian inference from the Kumaraswamy-Weibull distribution with applications to real data. International Journal of Contemporary Mathematical Sciences, 11(3): 119-129. https://doi.org/10.12988/ijcms.2016.51162

Mitnik PA (2013). New properties of the Kumaraswamy distribution. Communications in Statistics-Theory and Methods, 42(5): 741-755. https://doi.org/10.1080/03610926.2011.581782

Najarzadegan H, Alamatsaz MH, and Hayati S (2017). Truncated Weibull-G more flexible and more reliable than Beta-G distribution. International Journal of Statistics and Probability, 6(5): 1-17. https://doi.org/10.5539/ijsp.v6n5p1

Tierney L (1994). Markov chains for exploring posterior distributions. The Annals of Statistics, 22: 1701-1728. https://doi.org/10.1214/aos/1176325750

Varian HR (1975). A Bayesian approach to real estate assessment. In: Fienberg S and Zellner A (Eds.), Studies in Bayesian econometric and statistics in Honor of Leonard J. Savage: 195-208. North-Holland Pub., Amsterdam, Netherlands.

Zea LM, Silva RB, Bourguignon M, Santos AM, and Cordeiro GM (2012). The beta exponentiated Pareto distribution with application to bladder cancer susceptibility. International Journal of Statistics and Probability, 1: 8-19. https://doi.org/10.5539/ijsp.v1n2p8