Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

Numerical higher-order Runge-Kutta methods in transient and damping analysis

A. G. Shaikh [1,*], U. Keerio [2], Wajid Shaikh [3], A. H. Sheikh [4]

[1]*Department of BS & RS, QUEST, Nawabshah, Pakistan*
[2]*Department of Electrical Engineering, QUEST, Nawabshah, Pakistan*
[3]*Department of Mathematics & Statistics, QUEST, Nawabshah, Pakistan*
[4]*Institute of Business Management (IOB), Karachi, Pakistan*

A R T I C L E I N F O

Article history:
Received 23 July 2021
Received in revised form
31 January 2022
Accepted 18 July 2022

Keywords:
RLC circuit
Numerical methods
Runge-Kutta
MATLAB
Damping

A B S T R A C T

Transient analysis of an RLC circuit (or LCR circuit) comprising of a resistor, an inductor, and a capacitor are analyzed. Kirchhoff's voltage and current laws were used to generate equations for voltages and currents across the elements in an RLC circuit. From Kirchhoff's law, the resulting first-order and second-order differential equations, The different higher-order Runge-Kutta methods are applied with MATLAB simulations to check how changes in resistance affect transient which is transitory bursts of energy induced upon power, data, or communication lines; characterized by extremely high voltages that drive tremendous amounts of current into an electrical circuit for a few millionths, up to a few thousandths, of a second, and are very sensitive as well important their critical and careful analysis is also very important. The Runge-Kutta 5th and Runge-Kutta 8th order methods are applied to get nearer exact solutions and the numerical results are presented to illustrate the robustness and competency of the different higher-order Runge-Kutta methods in terms of accuracy.

1. Introduction

The transient analysis in the circuits and how the basic circuit elements like a resistor, capacitor, and inductor behave in the transient is of great importance (Das, 2010). Whenever we switch on the power supply in the circuit or turn off the supply or anything in the circuit changes abruptly, then the circuit takes some time to respond to this new condition and attain new steady state values. During the circuit operation, if anything or any parameter changes abruptly due to some surge or spike, then due to these abrupt changes, there could be a circuit failure or components failure. In addition, to investigate this failure one needs to look into the transient analysis that how voltage and current across the circuit elements changes during this transient. So, if we do the transient analysis then we can design the circuit in such a way that it can withstand such abrupt changes. Apart from circuit stability and failure analysis, in switching applications also this transient analysis is quite helpful. Transient normally results in changing the state of the components of an electrical circuit. It is very difficult for the capacitor voltage and the inductor current in an electrical circuit to assume a new steady state value. Transient analysis is very important since it can be used in analyzing the performance of any electrical circuit (Kee and Ranom, 2018). Thus, for an electrical current or voltage flowing through an electrical circuit, there can be various forms of the voltage or current. Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL) are normally in the form of differential equations rather than algebraic equations, these differential equations are not easily solved analytically when the order is high and complex. The numerical or approximate methods are one of the best techniques in solving almost all mathematical equations. The Runge-Kutta method was by far and away, the world's most popular numerical method for over 100 years, therefore the use of particular iterative methods depends on their efficiency. The efficiency of iterative methods depends on the stability, and cost in terms of time, suitability, and accuracy (Kafle et al., 2021). Without the use of the most accurate method, one might not be able to get an accurate solution and this might affect further

* Corresponding Author.
Email Address: agshaikh@quest.edu.pk (A. G. Shaikh)
https://doi.org/10.21833/ijaas.2022.10.020
Corresponding author's ORCID profile:
https://orcid.org/0000-0001-7367-993X

decisions based on the results (Suhag, 2013). Therefore, in this paper, three different variants of the same family the RK4, RK5, and RK8 methods are used in analyzing the transient behaviors and the damping factor in an RLC circuit taken into consideration and determine which method is the best in terms of accuracy and convergence analyzed.

2. Runge-Kutta method

The Runge-Kutta method is stable which means it doesn't exceed or diverge away from the exact solution, it retains the same shape and it does not diverge, on the contrary, the Euler method is unstable because it tends to do the kind of thing where it diverges (Ahmadianfar et al., 2021). The good thing about the Runge-Kutta study is very accurate and it can be used to solve virtually any ordinary differential equation even a nonlinear differential equation. We have seen in Euler's method if we want a more accurate solution or a better approximation of the solution, what do we need to do? We have to use a smaller steps value of h or in Taylor's method we want a more accurate solution, what do we need to do? We have to go for higher-order terms. The higher-order terms mean higher-order derivatives of a function. But whether we are decreasing step size or we are calculating the higher-order derivatives in both cases we need to do more calculations, and computational complexity will increase. So in RK methods, we attempt to obtain greater accuracy and at the same time avoid the need for calculation of higher derivatives or with a smaller step size.

2.1. Runge-Kutta of 4th order

Runge-Kutta 4th order method (sometimes known as RK4) is reasonably simple and robust. It is one of the simplest methods to remember which makes it very popular, in old days (Williamson, 1980; Ashgi et al., 2021).

2.1.1. Construction of the formulae

Although all formulae of this paper hold for systems of differential equations, for the sake of brevity the initial value problem for a single second-order differential special form,

$$y'' = f(x, y) \quad (1)$$

$$y(x_0) = y_0, \quad \text{and} \quad y'(x_0) = y'_0$$

$$y_{i+1} = y_i + \frac{1}{6}(R1 + 2(R2 + R3) + R4) + O(h^5) \quad (2)$$

where,

$$\left.\begin{aligned} R1 &= hf(x_n, y_n) \\ R2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}R1) \\ R3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}R2) \\ R4 &= hf(x_n + h, y_n + R3) \end{aligned}\right\} \quad (3)$$

2.2. Runge-Kutta of 5th order

It is not always better just to use the higher algorithms there are several tradeoffs one of the trade-offs is the amount of time. Butcher's method or 5th order Runge-Kutta might be a little bit overkill and so that is something to consider, and significantly better than several popular methods, but then we do have some computational overhead as well for doing the 5th order which is given by;

$$x_{i+1} = x + \frac{1}{90}(7R1 + 35R3 + 12R4 + 32R5 + 7R6) + O(h^6) \quad (4)$$

where,

$$\left.\begin{aligned} R1 &= hf(x_n, y_n) \\ R2 &= hf(x_n + \tfrac{2}{5}h, y_n + \tfrac{2}{5}R1) \\ R3 &= hf(x_n + \tfrac{1}{4}h, y_n + \tfrac{11}{64}R1 + \tfrac{5}{64}R2) \\ R4 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{3}{16}R1 + \tfrac{5}{16}R2) \\ R5 &= hf(x_n + \tfrac{3}{4}h, y_n + \tfrac{9}{32}R1 - \tfrac{27}{32}R2 + \tfrac{3}{4}R3 + \tfrac{9}{16}R4) \\ R6 &= hf(x_n + h, y_n - \tfrac{9}{28}R1 + \tfrac{35}{28}R2 - \tfrac{12}{7}R4 + \tfrac{8}{7}R5) \end{aligned}\right\} \quad (5)$$

2.3. Runge-Kutta of 8th order

In general, the methods of an order higher than 7 have an added computational cost which is usually not outweighed by order, given the tolerances chosen. One reason for this is that the coefficient choices for lower order methods are more optimized (they have small "principle truncation error coefficients," which matter more when we are not asymptotically small), the Runge-Kutta 8th order given by;

$$\left.\begin{aligned} x_{i+1} &= x_i + \frac{h}{840}(41R1 + 27R4 + 272R5 + 27R6 + 216R7 + 216R9 + 41R10) \\ y_{i+1} &= y_i + \frac{h}{840}(41M1 + 27M4 + 272M5 + 27M6 + 216M7 + 216M9 + 41M10) \end{aligned}\right\} \quad (6)$$

whereas the different stages of various orders of Runge-Kutta are presented. Table 1 shows a minimum number of stages for various orders.

2.4. Transient and RLC circuit

The transient response of RLC circuit Fig. 1 with external DC excitations, the transient is generated in electrical circuits due to abrupt changes in the operating conditions when energy storage elements like inductor or capacitor are present, this transient response is the dynamic response during the initial phase before the steady-state response is achieved when such abrupt changes are applied, so to obtain this transient response of such circuits we have to solve the differential equations, which are governing equations representing the electrical behavior of the circuit, a circuit having a single energy storage element either capacitor or an inductor is called a single order circuit and its governing differential equation is called a first order differential equation a circuit having both inductor and capacitor is called a second order circuit and its governing equation is called a second order differential equation so the variables of these differential equations are currents and voltages in the circuit function of time. Where C is Capacitance and $V_c(t)$ is a voltage across capacitance then the KVL equation for current is expressed in Eq. 7,

$$v = v_R + v_C + v_L \tag{7}$$

Table 1: Minimum number of stages for various orders

Order	2	3	4	5	6	7	8
Minimum Stages	2	3	4	6	7	9	11

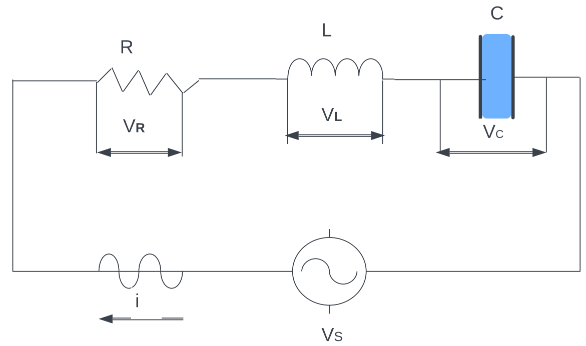

Fig. 1: RLC circuit with capacitor, induction, and resistance

Since the RLC circuit is described as a second-order differential equation, the voltage across the 2nd order RLC circuit according to Suhag (2013) is given by Eq. 8, where V_{in} is the input voltage.

$$L\,C\frac{d^2V_c(t)}{dt^2} + R * C\frac{dV_c(t)}{dt} + V_c(t) = V_{in} \tag{8}$$

$$v = v_c + v_p \tag{9}$$

where v_c is the transient response of the circuit, and v_p is a steady-state response of a circuit. For v_c:

$$v_c = [C_1\cos(\omega_d t) + C_2\sin(\omega_d t)] * e^{-\sigma * t} \tag{10}$$

$$v_p = \frac{\frac{12}{LC}}{\omega_n^2} = \frac{0.01*0.000001*0.12}{(10000)^2} = 12\ v \tag{11}$$

$$v_c = 12 + [A\cos(\omega_d t) + B\sin(\omega_d t)] * e^{-\sigma * t} \tag{12}$$

$$v_c(0) = 12 + [A\cos(0) + B\sin(0)] * e^0$$

$$\Rightarrow \quad B = \frac{-12\sigma}{\omega_d}$$

$$i(t) = C * \frac{dv_c}{dt} = \frac{d}{dt}[12 + [A\cos(\omega_d t) + B\sin(\omega_d t)] * e^{-\sigma * t}]$$

$$i(0) = C * [A(-\sigma) + B(\omega_d)] \tag{13}$$

$$\omega_n = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{0.01*0.000001}} = 10000$$

$$\sigma = \xi * \omega_n = 0.5*10000 = 5000$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 10000 * \sqrt{1-0.5^2} = 8660.25$$

$$B = \frac{-12*5000}{8660.25} = -6.93$$

$$v = 12 - [12\cos(8660.25 * t) + 6.93\sin(8660.25 * t)] * e^{-5000*t} \tag{14}$$

Again by considering homogenous form circuit differential equation,

$$\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = 0. \tag{15}$$

The characteristic equation will be $D^2 + \frac{R}{L}D + \frac{1}{LC}v_c = 0$,

$$D = \frac{-R}{2L} \pm \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}$$

letting $D_1 = -\alpha + \beta$, $\qquad D_2 = -\alpha - \beta$.

Notice that LC induces the natural oscillation of the circuit, which means they tend to induce the oscillation inside the circuit, while R has a tendency to damp out the oscillation, depending on the value of LC, the response of the circuit can be found, say the LC created natural oscillation by $\omega = \frac{1}{\sqrt{LC}}$, which is the frequency of oscillation known as natural frequency. Earlier we assumed that, $\alpha = \frac{R}{2L}$ and known as the damping coefficient of the circuit because this coefficient decides how well the circuit damping the oscillation, a part of this ratio $\frac{\alpha}{\omega} = \xi$, or $\xi = \frac{R}{2}\sqrt{\frac{C}{L}}$ known as the damping factor of the circuit, so this is known as normalizing the damping coefficients, the value of the ξ decides different oscillations. The transient response or type that a circuit exhibit is dependent on the value of the damping factor. The damping factor is the amount by which the oscillation of a system gradually decreases with time (Kee and Ranom, 2018). The following are various characteristics of the damping factor:

1. If $\xi > 1$, the system is overdamped.
2. If $\xi = 1$, the system is critically damped.
3. If $\xi < 1$, the system is underdamped.

Since the RLC circuit is described as a second-order differential equation, the voltage across 2nd order RLC circuit according, which is replaced by 1st order differential Eq. 16,

$$\frac{di}{dt} + \frac{R}{L}i + \frac{1}{LC}v_c = \frac{v_{in}}{LC} \tag{16}$$

where, $\frac{dv_c}{dt} = i$ we will discuss a few cases, therefore,

(Case I), R=100 Ω

$$\alpha = \frac{R}{2L} = \frac{100}{2*0.01} = 5000 \tag{17}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-2}}\sqrt{10^{-6}}} = 10000 \tag{18}$$

$$\frac{\alpha}{\omega_0} = \frac{5000}{10000} = 0.5$$

or,

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{100}{2}\sqrt{\frac{0.000001}{0.01}} = 0.5 \tag{19}$$

Eqs. 17, 18, and 19 yield $\xi = 0.5$. From the resulting damping factor value, the system is underdamped. Table 2 shows the voltage outcomes for an underdamped system for the RK4, RK5, and RK8 methods as time increases. From Table 2, it can be observed that the voltage obtained from RK8 is slightly better than RK4, and RK5 at each time t, whereas the numerical results of Table 2 also show that RK5 is better than RK4. Fig. 2 shows a plot of voltage against time for all RK orders used in this paper. From Fig. 2, it is observed that the RK8 method also converges faster than the RK4 and RK5 methods.

Table 2: Comparison of different variants of RK

Time(t)	Under-Damped, R = 100		
	RK_4	RK_5	RK_8
0.0000	0	0	0
0.0001	4.00000	4.08125	4.08362
0.0002	10.1875	10.1878	10.1920
0.0003	13.5790	13.4886	13.4909
0.0004	13.9329	13.8375	13.8369
0.0005	12.9353	12.8974	12.8953
0.0006	12.0123	12.0296	12.0280
0.0007	11.6574	11.6931	11.6927
0.0008	11.7241	11.7477	11.7482
0.0009	11.9107	11.9145	11.9151
0.0010	12.0332	12.0256	12.0259
0.0011	12.0560	12.0516	12.0516
0.0012	12.0349	12.0312	12.0310

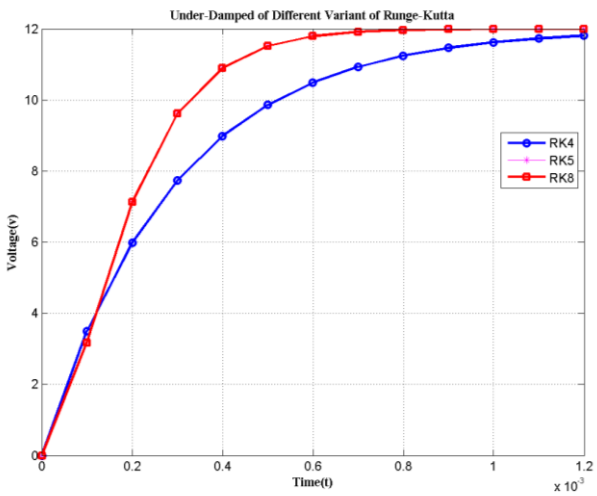

Fig. 2: Comparing different variants of RK4, RK5, and RK8

(Case II), R=200 Ω

$$\alpha = \frac{R}{2L} = \frac{200}{2*0.01} = 10000 \quad (21)$$

$$\omega_0 = 10000 \quad (22)$$

$$\frac{\alpha}{\omega_0} = \frac{10000}{10000} = 1.0 \quad (23)$$

Eqs. 21, 22, and 23 yield $\xi = 1.0$. Table 3, presents that as time increases the voltage also increases, also observed in classical RK4, the order is better and convergence faster than both RK5, and RK8 methods, whereas RK5 is better and convergences fast than RK8, as presented in Table 3, and Fig. 3.

(Case III), R=300 Ω

$$\alpha = \frac{R}{2L} = \frac{300}{2*0.01} = 15000.$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-2}}\sqrt{10^{-6}}} = 15000$$

$$\frac{\alpha}{\omega_0} = \frac{15000}{10000} = 1.5$$

Hence, the system is overdamped since the damping factor is greater than one. From Table 4, it can be observed; that voltage obtained from the RK4 order is slightly better than the RK5 and RK8 at each time t, whereas the numerical results of Table 4 also show both RK5 are better than RK8. Fig. 4 shows a plot of voltage against time for all RK orders used in this paper. From Fig. 4, it is observed that the RK4 method also converges faster than the RK5 and RK8.

Table 3: Comparison of different variants of RK

Time(t)	Critically-Damped, R=200		
	RK_4	RK_5	RK_8
0.0000	0	0	0
0.0001	3.50000	3.19375	3.16792
0.0002	7.31250	7.14167	7.12651
0.0003	9.67969	9.61611	9.60986
0.0004	10.9189	10.90313	10.9011
0.0005	11.5155	11.5154	11.5150
0.0006	11.7886	11.7919	11.7919
0.0007	11.9096	11.9124	11.9125
0.0008	11.9619	11.9637	1.19638
0.0009	11.9842	11.9852	1.19852
0.0001	11.9935	11.9940	1.19940
0.0011	11.9973	11.9976	1.19976
0.0012	11.9989	11.9990	1.19990

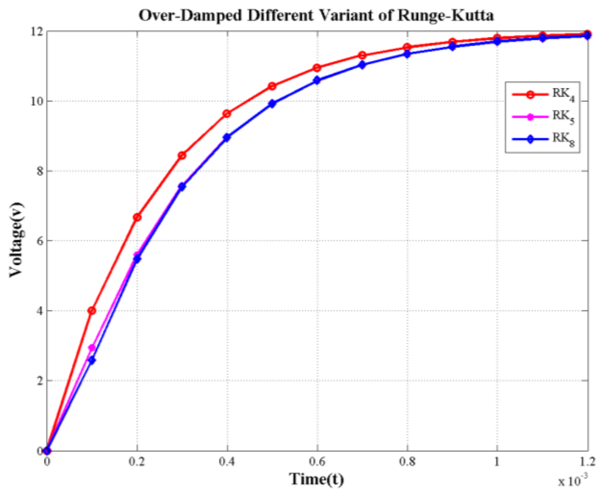

Fig. 3: Comparing different variants of RK4, RK5, and RK8

Table 4: Comparison of different variants of Runge-Kutta

Time(t)	Over-Damped, R=300		
	RK_4	RK_5	RK_8
0.0000	0	0	0
0.0001	4.00000	2.93125	2.58624
0.0002	6.68750	5.58731	5.47020
0.0003	8.48871	7.56657	7.53434
0.0004	9.69242	8.95971	8.95135
0.0005	1.04941	9.92129	9.91918
0.0006	11.0258	10.58032	10.05798
0.0007	11.3769	11.0308	11.0307
0.0008	11.6068	11.3384	11.3384
0.0009	11.7566	11.5485	11.5485
0.0010	11.8533	11.6918	11.6918
0.0011	11.9149	11.7897	11.7897
0.0012	11.9536	11.8564	11.8564

Table 5 shows a comparison of the absolute error of different methods of Runge-Kutta and Fig. 5 shows comparative of different Runge-Kutta methods in terms of absolute errors.

3. Conclusion

An important conclusion to be drawn from the overall numerical, as well as graphical and bar chart tests is that the construction of high-order Runge-Kutta formulae is worthwhile, that RK4 gives comparatively superiority, over RK5, and RK8 numerical results as well as graphical results, presented, whereas in terms of error the RK8 is superior from RK5, and RK4, numerical results, and graphical results confirm it for this problem considered in this paper. It is therefore recommended that the classical RK4 method is better to use, but in a few cases where accuracy is necessary, and most important having sensitive data, the preference is given to the higher-order RK8 to meet the required accuracy, Table 5, shows the superiority of the RK8 method over RK4, and RK8 in terms of error.

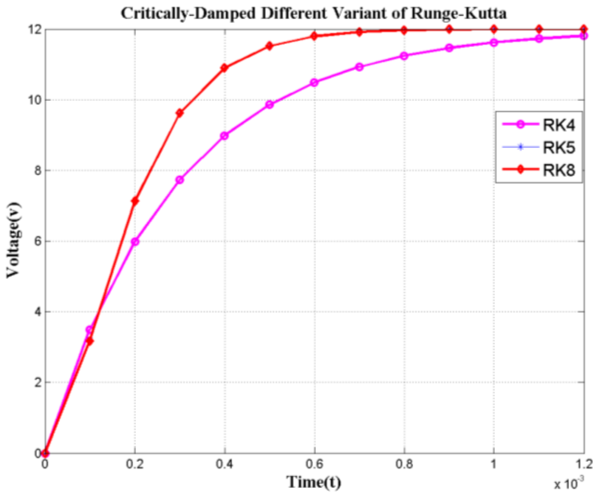

Fig. 4: Comparing different variants of RK4, RK5, and RK8

Table 5: Comparison of absolute error of different methods of Runge-Kutta

h	T	Y	Abs(ERRrk4)	Abs(ERRrk5)	Abs(ERRrk8)
0.0001	0	0	0	0	0
0.0001	0.0001	4.082766857	0.082766857	0.001516857	0.000858584
0.0001	0.0002	10.19245135	3.525784683	0.004668146	0.000448199
0.0001	0.0003	13.4920466	5.047602152	0.003476465	0.001121556
0.0001	0.0004	13.83754969	4.207920057	2.34594E-05	0.000659883
0.0001	0.0005	12.89522506	2.475471972	0.002174753	0.000105333
0.0001	0.0006	12.02755479	1.081052732	0.002072006	0.000470801
0.0001	0.0007	11.69232023	0.394652197	0.000770517	0.000367685
0.0001	0.0008	11.74805964	0.216280945	0.000332522	9.09678E-05
0.0001	0.0009	11.91519077	0.227338309	0.000661786	9.66645E-05
0.0001	0.001	12.02603266	0.234131018	0.000425087	0.000126164
0.0001	0.0011	12.05165847	0.190390708	6.90512E-05	6.46423E-05
0.0001	0.0012	12.03102117	0.123509334	0.000129127	9.39348E-07

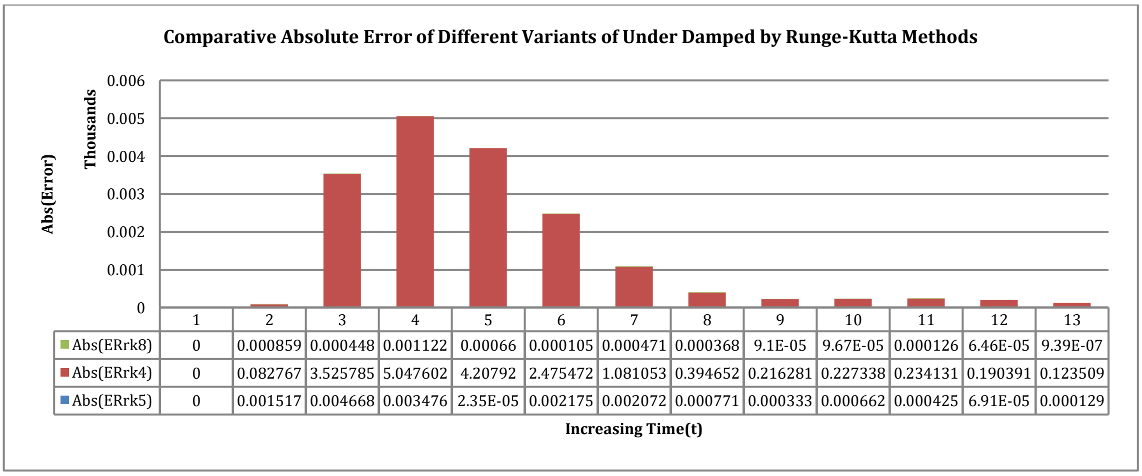

Fig. 5: Comparison of different Runge-Kutta methods in terms of absolute errors

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

References

Ahmadianfar I, Heidari AA, Gandomi AH, Chu X, and Chen H (2021). RUN beyond the metaphor: An efficient optimization algorithm based on Runge Kutta method. Expert Systems with Applications, 181: 115079. https://doi.org/10.1016/j.eswa.2021.115079

Ashgi R, Pratama MA, and Purwani S (2021). Comparison of numerical simulation of epidemiological model between Euler method with 4th Order Runge Kutta method. International Journal of Global Operations Research, 2(1): 37-44. https://doi.org/10.47194/ijgor.v2i1.67

Das JC (2010). Transients in electrical systems: Analysis, recognition, and mitigation. McGraw-Hill, New York, USA.

Kafle J, Thakur BK, and Bhandari IB (2021). Application of numerical methods for the analysis of damped parallel RLC circuit. Journal of Institute of Science and Technology, 26(1): 28-34. https://doi.org/10.3126/jist.v26i1.37814

Kee TA and Ranom R (2018). Comparison of numerical techniques in solving transient analysis of electrical circuits. ARPN Journal of Engineering and Applied Sciences, 13(1): 314-320.

Suhag A (2013). Transient analysis of electrical circuits using Runge-Kutta method and its application. International Journal of Scientific and Research Publications, 3: 486-490.

Williamson JH (1980). Low-storage Runge-Kutta schemes. Journal of Computational Physics, 35(1): 48-56. https://doi.org/10.1016/0021-9991(80)90033-9