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Fuzzy soft set theory applied to commutative ideals of BCK-algebras

G. Muhiuddin [1,*], D. Al-Kadi [2], Habib Harizavi [3], Amjad S. Albjedi [1]

[1]*Department of Mathematics, University of Tabuk, Tabuk, Saudi Arabia*
[2]*Department of Mathematics and Statistic, Taif University, Taif, Saudi Arabia*
[3]*Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran*

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ABSTRACT

In the present paper, we apply the fuzzy soft set theory to commutative ideals of BCK-algebras. In fact, the notion of fuzzy soft commutative ideals over BCK-algebras is introduced, and related properties are investigated. Relations between fuzzy soft ideals and fuzzy soft commutative ideals are discussed, and conditions for a fuzzy soft ideal to be a fuzzy soft commutative ideal are provided. The "AND" operation, "extended intersection" and "union" of fuzzy soft (commutative) ideals are studied. Furthermore, characterizations of fuzzy soft (commutative) ideals are considered.

1. Introduction

The concept of the soft set has been introduced by Molodtsov (1999) to reduce the difficulties that appear with using the classical theoretical approaches when dealing with uncertainties. Soft set theory is rich with applications some have been pointed out by Molodtsov (1999) and Maji et al. (2002). The work on soft sets has been growing rapidly. Maji et al. (2003) studied several operations on the theory of soft sets.

Also, Maji et al. (2001) generalized the standard notion of soft sets to the fuzzy soft sets and presented an application in a decision-making problem. Moreover, some results were presented by Roy and Maji (2007) on an application of fuzzy soft sets in decision-making problems.

The theory of fuzzy soft set has been applied to groups, BCK/BCI-algebras in Aygünoǧlu and Aygün (2009) and Jun et al. (2010), respectively. Al-Roqi et al. (2017) applied the notion of soft set theory to the filters in R0-algebras. Many authors have studied the soft set theory and fuzzy set theory on various aspects (Jun et al., 2017a; 2014; 2017b; 2016; Muhiuddin et al., 2017a; 2014; 2020a; 2020b; 2019a; Muhiuddin and Al-Roqi, 2015a; 2014; Muhiuddin and Mahboob, 2020). The ideal theory in algebraic structures has been studied in Muhiuddin and Al-Kadi (2021), Muhiuddin et al. (2020d; 2021), Senapati et al. (2019), and Talee et al. (2020). Some important generalizations of fuzzy sets were applied in BCK/BCI-algebras (Muhiuddin and Aldhafeeri, 2019; Muhiuddin and Al-Roqi, 2016; 2018; Muhiuddin and Jun, 2019; Muhiuddin et al., 2017b; 2017c; 2019b; 2019c; 2019d; 2020c; Muhiuddin, 2019).

In the present analysis, we investigate further properties of fuzzy soft ideals over BCK/BCI-algebras. We introduce the notion of fuzzy soft commutative ideals over BCK-algebras, and investigate related properties. We provide relations between fuzzy soft ideals and fuzzy soft commutative ideals. We discuss conditions for a fuzzy soft ideal to be a fuzzy soft commutative ideal. We consider the "AND" operation, "extended intersection" and "union" of fuzzy soft (commutative) ideals. We give characterizations of fuzzy soft (commutative) ideals.

2. Basic results on BCK/BCI-algebras

A *BCI-algebra* is a triple $(\Omega;*,0)$ of type (2,0) which satisfies the following conditions:

- $(\forall\omega_1,\omega_2,\omega_3 \in \Omega)\ (((\omega_1 * \omega_2) * (\omega_1 * \omega_3)) * (\omega_3 * \omega_2) = 0)$,
- $(\forall\omega_1,\omega_2 \in \Omega)\ ((\omega_1 * (\omega_1 * \omega_2)) * \omega_2 = 0)$,
- $(\forall\omega \in \Omega)\ (\omega * \omega = 0)$,
- $(\forall\omega_1,\omega_2 \in \Omega)\ (\omega_1 * \omega_2 = 0, \omega_2 * \omega_1 = 0 \Rightarrow \omega_1 = \omega_2)$

A *BCK-algebra* is a BCI-algebra Ω that satisfies the identity:

* Corresponding Author.
Email Address: chishtygm@gmail.com (G. Muhiuddin)
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Corresponding author's ORCID profile:
https://orcid.org/0000-0002-5596-5841

• $(\forall \omega \in \Omega)\ (0 * \omega = 0)$.

any BCK-algebra Ω satisfies the following identities:

• $(\forall \omega \in \Omega)\ (\omega * 0 = \omega)$,
• $(\forall \omega_1, \omega_2, \omega_3 \in \Omega)\ (\omega_1 \leq \omega_2 \Rightarrow \omega_1 * \omega_3 \leq \omega_2 * \omega_3, \omega_3 * \omega_2 \leq \omega_3 * \omega_1)$,
• $(\forall \omega_1, \omega_2, \omega_3 \in \Omega)\ ((\omega_1 * \omega_2) * \omega_3 = (\omega_1 * \omega_3) * \omega_2)$,
• $(\forall \omega_1, \omega_2, \omega_3 \in \Omega)\ ((\omega_1 * \omega_3) * (\omega_2 * \omega_3) \leq \omega_1 * \omega_2)$

where $\omega_1 \leq \omega_2$ if and only if $\omega_1 * \omega_2 = 0$.

An *ideal* J of a BCK/BCI-algebra Ω is a subset of Ω which contains 0 and satisfies the condition:

$$(\forall \omega_1, \omega_2 \in \Omega)(\omega_1 * \omega_2 \in J, \omega_2 \in J \Rightarrow \omega_1 \in J) \quad (1)$$

A commutative ideal J of a BCK-algebra Ω is a subset of Ω which contains 0 and satisfies the following condition:

$$(\forall \omega_1, \omega_2, \omega_3 \in \Omega)\ \big((\omega_1 * \omega_2) * \omega_3 \in J, \omega_3 \in J \Rightarrow \omega_1 * (\omega_2 * (\omega_2 * \omega_1)) \in J\big) \quad (2)$$

Note that, in BCK-algebras, every commutative ideal is an ideal, but not the converse (Meng and Jun, 1994).

A fuzzy ideal ϑ of a BCK/BCI-algebra Ω is a fuzzy set which satisfies:

$$(\forall \omega \in \Omega)(\vartheta(0) \geq \vartheta(\omega)), \quad (3)$$
$$(\forall \omega_1, \omega_2 \in \Omega)(\vartheta(\omega_1) \geq \min\{\vartheta(\omega_1 * \omega_2), \vartheta(\omega_2)\}) \quad (4)$$

A fuzzy commutative ideal ϑ of a BCK-algebra Ω is a fuzzy set in which (3) is satisfied and,

$$(\forall \omega_1, \omega_2, \omega_3 \in \Omega)(\vartheta(\omega_1 * (\omega_2 * (\omega_2 * \omega_1))) \geq \min\{\vartheta((\omega_1 * \omega_2) * \omega_3), \vartheta(\omega_3)\}) \quad (5)$$

3. Basic results on (fuzzy) soft sets

Let $P(U)$ denotes the power set of an initial universe set U and E be a set of parameters. Consider $K \subset E$. Then the soft set over U is defined by Molodtsov as a pair (κ, K) where κ is a mapping given by $\kappa: K \to P(U)$ (see Molodtsov (1999) for illustration and examples).

Definition 3.1 (Maji et al., 2001)**:** Let $\mathrm{F}(U)$ denotes the set of all fuzzy sets in U. Then $(\tilde{\kappa}, K)$ is called a fuzzy soft set over U where $\tilde{\kappa}$ is a mapping given by $\tilde{\kappa}: K \to \mathrm{F}(U)$. For a parameter $k \in K$, we write $\tilde{\kappa}[k]$ for the fuzzy value set of k.

For two fuzzy soft sets $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ over a common universe U, the following are defined in Maji et al. (2001).

Definition 3.2: We say that $(\tilde{\kappa}, K)$ is a fuzzy soft subset of $(\tilde{\upsilon}, V)$, if it satisfies*:*

1. $K \subseteq V$,
2. $(\forall k \in K)(\ [\mathrm{k}]\ \textit{is a fuzzy subset of}\ [\mathrm{k}])$.

We write $(\tilde{\kappa}, K)\ \tilde{\subseteq}\ (\tilde{\upsilon}, V)$.

Definition 3.3: The "union" of $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ is the fuzzy soft set $(\tilde{\tau}, T)$ such that:

• $T = K \cup V$,
• for all $t \in T$,

$$\tilde{\tau}[t] = \begin{cases} \tilde{\kappa}[t] & if\ \ t \in\ K \setminus V, \\ \tilde{\upsilon}[t] & if\ \ t \in\ V \setminus K, \\ \tilde{\kappa}[t] \cup \tilde{\upsilon}[t] & if\ \ t \in\ K \cap\ V. \end{cases}$$

we write $(\tilde{\kappa}, K)\ \tilde{\cup}\ (\tilde{\upsilon}, V) = (\tilde{\tau}, T)$.

Definition 3.4: We define $(\tilde{\kappa}, K)$ *"AND"* $(\tilde{\upsilon}, V)$ by $(\tilde{\kappa}, K)\ \tilde{\wedge}\ (\tilde{\upsilon}, V) = (\tilde{\tau}, K \times V)$, where $\tilde{\tau}[k, v] = \tilde{\kappa}[k] \cap \tilde{\upsilon}[v]\ \textit{for all}\ (k, v) \in K \times V$. We write $(\tilde{\kappa}, K)\ \tilde{\wedge}\ (\tilde{\upsilon}, V)$.

For two soft sets $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ over a common universe U, the following operations are defined in Ali et al. (2009).

Definition 3.5: The "extended intersection" of $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ is the soft set $(\tilde{\tau}, T)$ where $T = K \cup V$ and for every $t \in T$:

$$\tilde{\tau}[t] = \begin{cases} \tilde{\kappa}[t] & \text{if}\ \ t \in K \setminus V, \\ \tilde{\upsilon}[t] & if\ \ t \in V \setminus K, \\ \tilde{\kappa}[t] \cap \tilde{\upsilon}[t] & if\ \ t \in K \cap V. \end{cases}$$

We write $(\tilde{\kappa}, K)\ \tilde{\cap}_{\mathrm{e}}\ (\tilde{\upsilon}, V) = (\tilde{\tau}, T)$.

Definition 3.6: The "restricted intersection" of $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ where $K \cap V \neq \emptyset$ is defined by $(\tilde{\kappa}, K)\ \tilde{\cap}_r\ (\tilde{\upsilon}, V) = (\tilde{\tau}, T)$, where $T = K \cap V$ and for all $t \in T$, $\tilde{\tau}[t] = \tilde{\kappa}[t] \cap \tilde{\upsilon}[t]$.

We write $(\tilde{\kappa}, K)\ \tilde{\cap}_{\mathrm{r}}\ (\tilde{\upsilon}, V)$.

4. Further properties of fuzzy soft ideals

In this section, Ω denotes BCK/BCI-algebras. Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over a BCK/BCI-algebra Ω then a fuzzy soft ideal is defined as follows:

Definition 4.1 (Jun et al., 2010)**:** A fuzzy soft ideal $(\tilde{\kappa}, K)$ over Ω is a fuzzy soft ideal over Ω based on all parameters. If there exists a parameter $k \in K$ such that $\tilde{\kappa}[k]$ is a fuzzy ideal of Ω, we say that $(\tilde{\kappa}, K)$ is a fuzzy soft ideal over Ω based on a parameter k.

Theorem 4.2: Let $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ be two fuzzy soft sets over a BCK/BCI-algebra Ω such that $(\tilde{\kappa}, K)\ \tilde{\subseteq}\ (\tilde{\upsilon}, V)$. *If* $(\tilde{\upsilon}, V)$ is a fuzzy soft ideal over Ω, then*:*

1. $(\forall \omega \in \Omega)(\forall k \in K)(\tilde{\upsilon}[k](0) \geq \tilde{\kappa}[k](\omega))$
2. $(\forall \omega_1, \omega_2 \in \Omega)(\forall k \in K)(\tilde{\upsilon}[k](\omega_1) \geq \min\{\tilde{\kappa}[k](\omega_1 * \omega_2), \tilde{\kappa}[k](\omega_2)\})$

Proof: Assume that $(\tilde{\upsilon}, \mathrm{V})$ is a fuzzy soft ideal over Ω. For any $k \in K$ and $\omega \in \Omega$, we have $\tilde{\upsilon}[k](0) \geq \tilde{\upsilon}[k](\omega) \geq \tilde{\kappa}[k](\omega)$, which proves (1). Now we have:

$$\tilde{\upsilon}[k](\omega_1) \geq \min\{\tilde{\upsilon}[k](\omega_1 * \omega_2), \tilde{\upsilon}[k](\omega_2)\} \geq \min\{\tilde{\kappa}[k](\omega_1 * \omega_2), \tilde{\kappa}[k](\omega_2)\}$$

for all $\omega_1, \omega_2 \in \Omega$ and $k \in K$, and so (2) is valid.

Question: For two fuzzy soft sets $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ over a *BCK/BCI-algebra* Ω, where $(\tilde{\kappa}, K) \mathrel{\tilde{\subseteq}} (\tilde{\upsilon}, V)$. *If* $(\tilde{\upsilon}, V)$ is a fuzzy soft ideal over Ω then is $(\tilde{\kappa}, K)$ a fuzzy soft ideal over Ω?

The answer to this question is negative as seen in the following example.

Example 4.3: Suppose that there are four private schools in the initial universe set U given by:

$U = \{s_0, s_1, s_2, s_3\}$

with the following binary operation:

$*$	s_0	s_1	s_2	s_3
s_0	s_0	s_0	s_0	s_0
s_1	s_1	s_0	s_0	s_1
s_2	s_2	s_2	s_0	s_2
s_3	s_3	s_3	s_3	s_0

Then $(U, *, s_0)$ is a BCK-algebra. Let a set of parameters $E = \{d, c, l, e\}$ be a set of the status of schools which stand for the parameters "leadership", "coast", "location" and "experience", respectively. Take $K = \{d\}$ and $V = \{d, l\}$. Let $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ be soft sets over U which are defined by:

$\tilde{\kappa}$	S_0	S_1	S_2	S_3
d	0.6	0.1	0.2	0.5

and

$\tilde{\upsilon}$	S_0	S_1	S_2	S_3
d	0.7	0.4	0.3	0.6
l	0.8	0.7	0.6	0.5

respectively. Then $(\tilde{\upsilon}, V)$ is a fuzzy soft ideal over U. But $(\tilde{\kappa}, K)$ is not a fuzzy soft ideal over U since,

$\tilde{\kappa}[d](s_1) = 0.1 < 0.2 = \min\{\tilde{\kappa}[d](s_1 * s_2), \tilde{\kappa}[d](s_2)\}$

Theorem 4.4: Let $(\tilde{\kappa}, K)$ be a fuzzy soft ideal over a BCK/BCI-algebra Ω. *If* V is a subset of K, then $(\tilde{\kappa}|_V, V)$ is a fuzzy soft ideal over a BCK/BCI-algebra Ω.

Proof: Straightforward.

Question: Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over a BCK/BCI-algebra Ω. *If* $(\tilde{\kappa}, K)$ is not a fuzzy soft ideal over a BCK/BCI-algebra Ω then does a subset V of K exists such that $(\tilde{\kappa}|_V, V)$ is a fuzzy soft ideal over a BCK/BCI-algebra Ω?

The answer to this question is positive as shown in the following example.

Example 4.5: Suppose there are five activities in the universe U, that is,

$U := \{shopping, walking, relaxing, working, swimming\}$

Let $\uplus$ be a terminology which chooses between two activities as follows:

shopping $\uplus$ u =shopping for all $u \in U$,

$$\text{walking} \uplus u = \begin{cases} shopping & \text{if } u \in \{walking, working, swimming\}, \\ walking & \text{if } u \in \{shopping, relaxing\}, \end{cases}$$

$$relaxing \uplus u = \begin{cases} shopping & \text{if } u \in \{relaxing, swimming\}, \\ relaxing & \text{if } u \in \{shopping, walking, working\}, \end{cases}$$

$$working \uplus u = \begin{cases} shopping & \text{if } u \in \{working, swimming\}, \\ working & \text{if } u \in \{shopping, walking, relaxing\}, \end{cases}$$

$$swimming \uplus u = \begin{cases} shopping & \text{if } u \in \{swimming\}, \\ relaxing & \text{if } u \in \{working\}, \\ working & \text{if } u \in \{relaxing\}, \\ swimming & \text{if } u \in \{shopping, walking\}. \end{cases}$$

Then $(U, \uplus, shopping)$ is a BCK-algebra (Jun et al., 2010). Consider a set of parameters:

$K := \{enjoyed, bored, frustrated, energized, stimulated\}$

Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over U. Then $\tilde{\kappa}[enjoyed]$, $\tilde{\kappa}$[bored], $\tilde{\kappa}[frustrated]$, $\tilde{\kappa}[energized]$, and $\tilde{\kappa}[stimulated]$ are fuzzy sets in U. We define them as follows:

$\tilde{\kappa}$	shopping	walking	relaxing	working	swimming
enjoyed	0.9	0.6	0.8	0.3	0.3
bored	0.8	0.7	0.4	0.6	0.4
frustrated	0.6	0.5	0.4	02	0.2
energized	0.1	0.2	0.3	0.4	0.5
stimulated	0.3	0.1	0.5	0.6	0.1

Then $(\tilde{\kappa}, K)$ is not a fuzzy soft ideal over U since $\tilde{\kappa}[energized]$ and $\tilde{\kappa}[stimulated]$ are not fuzzy ideals in U. But if we take,

$V := \{enjoyed, bored, frustrated\}$,

then $(\tilde{\kappa}|_V, V)$ is a fuzzy soft ideal over U.

Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over a BCK/BCI-algebra Ω and $t \in [0,1]$. For a parameter $k \in K$, consider the following sets:

$(\tilde{\kappa}, K)_k^{\geq t} := \{\omega \in \Omega | \tilde{\kappa}[k](\omega) \geq t\}$

and

$(\tilde{\kappa}, K)_t^{\gtrapprox} := \{\omega \in \Omega | \tilde{\kappa}[k](\omega) \geq t, for all k \in K\}$.

Obviously, $(\tilde{\kappa}, K)_t^{\gtrapprox} = \bigcap_{k \in K} (\tilde{\kappa}, K)_k^{\geq t}$.

Theorem 4.6: For a fuzzy soft set $(\tilde{\kappa}, K)$ over a BCK/BCI-algebra Ω, the following are equivalent:

1. $(\tilde{\kappa}, K)$ is a fuzzy soft ideal over Ω based on a parameter $k \in K$.
2. $(\tilde{\kappa}, K)_k^{\geq t}$ is an ideal of Ω, for all $t \in [0,1]$ with $(\tilde{\kappa}, K)_k^{\geq t} \neq \emptyset$.

Proof: Assume that $(\tilde{\kappa}, K)$ is a fuzzy soft ideal over Ω based on a parameter $k \in K$ and let $t \in [0,1]$ such that $(\tilde{\kappa}, K)_k^{\geq t} \neq \emptyset$. Obviously, $0 \in (\tilde{\kappa}, K)_k^{\geq t}$. Let $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 * \omega_2 \in (\tilde{\kappa}, K)_k^{\geq t}$ and $\omega_2 \in (\tilde{\kappa}, K)_k^{\geq t}$. Then $\tilde{\kappa}[k](\omega_1 * \omega_2) \geq t$ and $\tilde{\kappa}[k](\omega_2) \geq t$. It follows from (4) that $\tilde{\kappa}[k](\omega_1) \geq \min\{\tilde{\kappa}[k](\omega_1 * \omega_2), \tilde{\kappa}[k](\omega_2)\} \geq t$ and so that $\tilde{\kappa}[k](\omega_1) \geq t$. Hence $\omega_1 \in (\tilde{\kappa}, K)_k^{\geq t}$, and therefore $(\tilde{\kappa}, K)_k^{\geq t}$ is an ideal of Ω for all $t \in [0,1]$ with $(\tilde{\kappa}, K)_k^{\geq t} \neq \emptyset$.

Conversely, suppose that (2) is valid. It is sufficient to show that $\tilde{\kappa}[k]$ satisfies the following two conditions:

(i). $\tilde{\kappa}[k](0) \geq \tilde{\kappa}[k](\omega)$ for all $\omega \in \Omega$.
(ii). $\tilde{\kappa}[k](\omega) \geq \min\{\tilde{\kappa}[k](\omega_1 * \omega_2), \tilde{\kappa}[k](\omega_2)\}$, for all $\omega_1, \omega_2 \in \Omega$.

If (i) is not valid, then there exists $\omega_0 \in \Omega$ such that $\tilde{\kappa}[k](0) < \tilde{\kappa}[k](\omega_0)$. Hence $\omega_0 \in (\tilde{\kappa}, K)_k^{\geq t_0}$ where $t_0 = \tilde{\kappa}[k](\omega_0)$, and so $(\tilde{\kappa}, K)_k^{\geq t_0} \neq \emptyset$. But $0 \notin (\tilde{\kappa}, K)_k^{\geq t_0}$, which is a contradiction. Thus $\tilde{\kappa}[k](0) \geq \tilde{\kappa}[k](\omega)$, for all $\omega \in \Omega$. Assume that (ii) is false. Then,

$$\tilde{\kappa}[k](\omega) < \min\{\tilde{\kappa}[k](\omega * \omega'), \tilde{\kappa}[k](\omega')\}$$

for some $\omega, \omega' \in \Omega$. If we take $t_\omega = \min\{\tilde{\kappa}[k](\omega * \omega'), \tilde{\kappa}[k](\omega')\}$, then $\omega * \omega' \in (\tilde{\kappa}, K)_k^{\geq t_\omega}$ and $\omega' \in (\tilde{\kappa}, K)_k^{\geq t_\omega}$. Since $(\tilde{\kappa}, K)_k^{\geq t_\omega}$ is an ideal of Ω, it follows that $\omega \in (\tilde{\kappa}, K)_k^{\geq t_\omega}$ and so that $\tilde{\kappa}[k](\omega) \geq t_\omega = \min\{\tilde{\kappa}[k](\omega * \omega'), \tilde{\kappa}[k](\omega')\}$. This is a contradiction, and the proof is complete.

Corollary 4.7: A fuzzy soft set $(\tilde{\kappa}, K)$ over a BCK/BCI-algebra Ω is a fuzzy soft ideal over Ω if and only if $(\tilde{\kappa}, K)_t^{\geq}$ is an ideal of Ω with $(\tilde{\kappa}, K)_t^{\geq} \neq \emptyset$ for all $t \in [0,1]$.

Theorem 4.8: If $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ are two fuzzy soft ideals over a BCK/BCI-algebra Ω, then the "extended intersection" of $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ is a fuzzy soft ideal over a BCK/BCI-algebra Ω.

Proof: Let $(\tilde{\kappa}, K) \mathbin{\tilde{\cap}_e} (\tilde{\upsilon}, V) = (\tilde{\tau}, T)$ be the "extended intersection" of $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$. Then $T = K \cup V$. For any $t \in T$, if $t \in K \setminus V$ (resp. $t \in V \setminus K$) then $\tilde{\tau}[t] = \tilde{\kappa}[t]$ (resp. $\tilde{\tau}[t] = \tilde{\upsilon}[t]$) is a fuzzy ideal. If $K \cap V \neq \emptyset$, then $\tilde{\tau}[t] = \tilde{\kappa}[t] \cap \tilde{\upsilon}[t]$ is a fuzzy ideal for all $t \in K \cap V$ since the intersection of two fuzzy ideals is a fuzzy ideal. Therefore $(\tilde{\tau}, T)$ is a fuzzy soft ideal over a BCK/BCI-algebra Ω.

The following two results follow directly from Theorem 4.8.

Corollary 4.9: Let $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, K)$ be two fuzzy soft ideals over a BCK/BCI-algebra Ω. Then the "extended intersection" $(\tilde{\kappa}, K) \mathbin{\tilde{\cap}_e} (\tilde{\upsilon}, K)$ is a fuzzy soft ideal over Ω.

Corollary 4.10: The "restricted intersection" of two fuzzy soft ideals is a fuzzy soft ideal.

Theorem 4.11: Let $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ be two fuzzy soft ideals over a BCK/BCI-algebra Ω. If $K \cap V = \emptyset$, then the "union" $(\tilde{\kappa}, K) \mathbin{\tilde{\cup}} (\tilde{\upsilon}, V)$ is a fuzzy soft ideal over a BCK/BCI-algebra Ω.

Proof: Since K and V are disjoint, then for any $t \in T$ either $t \in K \setminus V$ or $t \in V \setminus K$, by means of Definition 3.3. If $t \in K \setminus V$, then $\tilde{\tau}[t] = \tilde{\kappa}[t]$ is a fuzzy ideal in a BCK/BCI-algebra Ω because $(\tilde{\kappa}, K)$ is a fuzzy soft ideal over a BCK/BCI-algebra Ω. If $t \in V \setminus K$, then $\tilde{\tau}[t] = \tilde{\upsilon}[t]$ is a fuzzy ideal in a BCK/BCI-algebra Ω because $(\tilde{\upsilon}, V)$ is a fuzzy soft ideal over a BCK/BCI-algebra Ω. Hence $(\tilde{\tau}, T) = (\tilde{\kappa}, K) \mathbin{\tilde{\cup}} (\tilde{\upsilon}, V)$ is a fuzzy soft ideal over a BCK/BCI-algebra Ω.

5. Fuzzy soft commutative ideals

In this section, Ω denotes a BCK-algebra unless otherwise is specified.

Definition 5.1: Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over a BCK-algebra Ω. If there exists a parameter $k \in K$ such that $\tilde{\kappa}[k]$ is a fuzzy commutative ideal of Ω, that is, the following assertions are valid:

$$\tilde{\kappa}[k](0) \geq \tilde{\kappa}[k](\omega), \quad (6)$$
$$\tilde{\kappa}[k](\omega_1 * (\omega_2 * (\omega_2 * \omega_1))) \geq \min\{\tilde{\kappa}[k]((\omega_1 * \omega_2) * \omega_3), \tilde{\kappa}[k](\omega_3)\} \quad (7)$$

for all $\omega_1, \omega_2, \omega_3 \in \Omega$, we say that $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over Ω based on a parameter k. If $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over Ω based on all parameters, then $(\tilde{\kappa}, K)$ is called a fuzzy soft commutative ideal over Ω.

Example 5.2: Let:

$$U := \{monday, tuesday, wednesday, thursday\}$$

be an initial universe, and consider the terminology $\Cup$ which chooses between the days as follows:

$$monday \Cup u = monday for all u \in U,$$
$$tuesday \Cup u = \begin{cases} monday & if \;\; u \in \{tuesday, wednesday\}, \\ tuesday & if \;\; u \in \{monday, thursday\}, \end{cases}$$
$$wednesday \Cup u = \begin{cases} wednesday & if \;\; u \in \{monday, thursday\}, \\ tuesday & if \;\; u \in \{tuesday\}, \\ monday & if \;\; u \in \{wednesday\}, \end{cases}$$
$$thursday \Cup u = \begin{cases} monday & if \;\; u = thursday, \\ thursday & if \;\; u \in \{monday, tuesday, wednesday\}. \end{cases}$$

Then $(U, \Cup, monday)$ is a BCK-algebra. Consider a set of parameters $K := \{happy, angry, calm\}$. Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over U. Then $\tilde{\kappa}[happy]$, $\tilde{\kappa}[angry]$, and $\tilde{\kappa}[calm]$ are fuzzy sets in U. We define them as follows:

$\tilde{\kappa}$	Monday	Tuesday	Wednesday	Thursday
happy	0.5	0.3	0.3	0.4
angry	0.9	0.2	0.2	0.2
calm	0.7	0.5	0.5	0.2

Then $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over U based on parameters "happy", "angry" and "calm".

Theorem 5.3: For any BCK-algebra, every fuzzy soft commutative ideal (based on a parameter) is a fuzzy soft ideal (based on the same parameter).

Proof: Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over a BCK-algebra Ω. Assume that $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over Ω based on a parameter $k \in K$. Then,

$$\begin{aligned}&\min\{\tilde{\kappa}[k](\omega_1 * \omega_3), \tilde{\kappa}[k](\omega_3)\}\\ &\quad= \min\{\tilde{\kappa}[k]((\omega_1 * 0) * \omega_3), \tilde{\kappa}[k](\omega_3)\}\\ &\quad\leq \tilde{\kappa}[k](\omega_1 * (0 * (0 * \omega_1)))\\ &\quad= \tilde{\kappa}(\omega_1)\end{aligned}$$

for all $\omega_1, \omega_3 \in \Omega$. Therefore $(\tilde{\kappa}, K)$ is a fuzzy soft ideal over Ω based on a parameter $k \in K$.

The converse of Theorem 5.3 is not true in general as shown in the example given next.

Example 5.4: Let:

$$U := \{monday, tuesday, wednesday, thursday, friday\}$$

be an initial universe, and consider a terminology Ⓢ that chooses between days in the following way:

$monday$ Ⓢ $u = monday for all u \in U$,

$tuesday$ Ⓢ $u = \begin{cases} monday & if \;\; u \in \{tuesday, thursday, friday\}, \\ tuesday & if \;\; u \in \{monday, wednesday\}, \end{cases}$

$wednesday$ Ⓢ $u = \begin{cases} wednesday & if \;\; u \in \{monday, tuesday\}, \\ monday & if \;\; u \in \{wednesday, thursday, friday\}, \end{cases}$

$thursday$ Ⓢ $u = \begin{cases} monday & if \;\; u \in \{thursday, friday\}, \\ thursday & if \;\; u \in \{monday, tuesday, wednesday\}, \end{cases}$

$friday$ Ⓢ $u = \begin{cases} friday & if \;\; u \in \{monday, tuesday, wednesday\}, \\ thursday & if \;\; u \in \{thursday\}, \\ monday & if \;\; u \in \{friday\}. \end{cases}$

Then $(U$, Ⓢ, $monday)$ is a BCK-algebra. Consider a set of parameters,

$$K := \{happy, angry, calm\}$$

Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over U. Then $\tilde{\kappa}[happy]$, $\tilde{\kappa}[angry]$, and $\tilde{\kappa}[calm]$ are fuzzy sets in U. We define them as follows:

$\tilde{\kappa}$	Monday	Tuesday	Wednesday	Thursday	Friday
happy	0.7	0.5	0.3	0.3	0.3
angry	0.8	0.4	0.6	0.4	0.4
calm	0.6	0.5	0.5	0.2	0.2

Then $(\tilde{\kappa}, K)$ is a fuzzy soft ideal over U based on parameters "happy", "angry" and "calm". But $(\tilde{\kappa}, K)$ is not a fuzzy soft commutative ideal over U based on parameters "happy" and "angry" since,

$\min\{\tilde{\kappa}[happy]((wednesday$ Ⓢ $thursday)$ Ⓢ $tuesday), \tilde{\kappa}[happy](tuesday)\}$
$= \tilde{\kappa}[happy](tuesday)$
$= 0.5$
$> \tilde{\kappa}[happy](wednesday$ Ⓢ $(thursday$ Ⓢ $(thursday$ Ⓢ $wednesday)))$
$= \tilde{\kappa}[happy](wednesday)$
$= 0.3$

and

$\min\{\tilde{\kappa}[angry]((tuesday$ Ⓢ $thursday)$ Ⓢ $wednsday), \tilde{\kappa}[angry](wednsday)\}$
$= \tilde{\kappa}[angry](wednsday)$
$= 0.6$
$> \tilde{\kappa}[angry](tuesday$ Ⓢ $(thursday$ Ⓢ $(thursday$ Ⓢ $tuesday)))$
$= \tilde{\kappa}[angry](tuesday)$
$= 0.4.$

Theorem 5.5: Let $(\tilde{\kappa}, K)$ be a fuzzy soft commutative ideal over a BCK-algebra *Ω*. *If* V is a subset of K, then $(\tilde{\kappa}|_V, V)$ is a fuzzy soft commutative ideal over *Ω.*

Proof: Direct to prove.

Question: Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over a BCK-algebra *Ω. If* $(\tilde{\kappa}, K)$ is not a fuzzy soft commutative ideal over a BCK-algebra *Ω,* then can we find a subset V of K such that $(\tilde{\kappa}|_V, V)$ is a fuzzy soft commutative ideal over a BCK-algebra *Ω*?

The answer is given by the following example.

Example 5.6: Consider BCK-algebra $(U, \uplus, shopping)$ in Example 4.5 and consider a set of parameters,

$$K := \{enjoyed, frustrated, bored\}$$

Let $(\tilde{\kappa}, K)$ be a fuzzy soft set over U. Then $\tilde{\kappa}[enjoyed]$, $\tilde{\kappa}[frustrated]$, and $\tilde{\kappa}[bored]$ are fuzzy sets in U. We define them as follows:

$\tilde{\kappa}$	shopping	walking	relaxing	working	swimming
enjoyed	0.9	0.9	0.5	0.3	0.3
frustrated	0.8	0.4	0.6	0.4	0.4
bored	0.7	0.7	0.4	0.5	0.4

Then $\tilde{\kappa}[frustrated]$ is a fuzzy ideal in U, but not a fuzzy commutative ideal in U since,

$$\begin{aligned}&\tilde{\kappa}[frustrated](walking \uplus (working \uplus (working \uplus walking)))\\&= \tilde{\kappa}[frustrated](walking) = 0.4 \ngeq 0.6\\&= \min\{\tilde{\kappa}[frustrated]((walking \uplus working) \uplus relaxing), \tilde{\kappa}[frustrated](relaxing)\}\\&= \{\tilde{\kappa}[frustrated](relaxing)\}.\end{aligned}$$

Thus $(\tilde{\kappa}, K)$ is not a fuzzy soft commutative ideal over U. But if we take,

$V := \{enjoyed, bored\}$,

then $(\tilde{\kappa}|_V, V)$ is a fuzzy soft commutative ideal over U.

Proposition 5.7: Every fuzzy soft commutative ideal $(\tilde{\kappa}, K)$ over a BCK-algebra Ω satisfies the following assertion:

$$(\forall \omega_1, \omega_2 \in \Omega)(\forall k \in K)$$
$$(\tilde{\kappa}[k](\omega_1 * \omega_2) \leq \tilde{\kappa}[k](\omega_1 * (\omega_2 * (\omega_2 * \omega_1)))) \quad (8)$$

Proof: If we take $\omega_3 = 0$ in (7) and use (a1) and (6), then

$\tilde{\kappa}[k](\omega_1 * (\omega_2 * (\omega_2 * \omega_1))) \geq \min\{\tilde{\kappa}[k]((\omega_1 * \omega_2) * 0), \tilde{\kappa}[k](0)\} = \tilde{\kappa}[k](\omega_1 * \omega_2)$ for all $\omega_1, \omega_2 \in \Omega$.

Theorem 5.8: If a fuzzy soft ideal $(\tilde{\kappa}, K)$ over a BCK-algebra Ω satisfies the condition (8), then $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over Ω.

Proof: Let $k \in K$ be a parameter and let $\omega_1, \omega_2, \omega_3 \in \Omega$. Then,

$$\tilde{\kappa}[k](\omega_1 * (\omega_2 * (\omega_2 * \omega_1))) \geq \tilde{\kappa}[k](\omega_1 * \omega_2) \geq \min\{\tilde{\kappa}[k]((\omega_1 * \omega_2) * \omega_3), \tilde{\kappa}[k](\omega_3)\}$$

Hence $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over Ω.

Lemma 5.9 (-3180): Let $(\tilde{\kappa}, K)$ be a fuzzy soft ideal over a BCK/BCI-algebra Ω. If Ω satisfies the inequality $\omega_1 * \omega_2 \leq \omega_3$, then:

$$\tilde{\kappa}[k](\omega_1) \geq \min\{\tilde{\kappa}[k](\omega_2), \tilde{\kappa}[k](\omega_3)\} \quad (9)$$

for all $\omega_1, \omega_2, \omega_3 \in \Omega$ and $k \in K$.

Theorem 5.10: In a commutative BCK-algebra Ω, every fuzzy soft ideal is a fuzzy soft commutative ideal.

Proof: Let $(\tilde{\kappa}, K)$ be a fuzzy soft ideal over a commutative BCK-algebra Ω. Note that:

$$\begin{aligned}&\left(\left(\omega_1 * (\omega_2 * (\omega_2 * \omega_1))\right) * ((\omega_1 * \omega_2) * \omega_3)\right) * \omega_3\\&= \left(\left(\omega_1 * (\omega_2 * (\omega_2 * \omega_1))\right) * \omega_3\right) * ((\omega_1 * \omega_2) * \omega_3)\\&\leq \left(\omega_1 * (\omega_2 * (\omega_2 * \omega_1))\right) * (\omega_1 * \omega_2)\\&= (\omega_1 * (\omega_1 * \omega_2)) * (\omega_2 * (\omega_2 * \omega_1)) = 0,\end{aligned}$$

that is, $(\omega_1 * (\omega_2 * (\omega_2 * \omega_1))) * ((\omega_1 * \omega_2) * \omega_3) \leq \omega_3$ for all $\omega_1, \omega_2, \omega_3 \in \Omega$. It follows from Lemma 5.9 that:

$$\tilde{\kappa}[k]\left(\omega_1 * (\omega_2 * (\omega_2 * \omega_1))\right) \geq \min\{\tilde{\kappa}[k]((\omega_1 * \omega_2) * \omega_3), \tilde{\kappa}[k](\omega_3)\}.$$

Thus (7) holds, and the proof is complete.

Lemma 5.11 (Jun et al., 2010)**:** Let Ω be a BCK/BCI-algebra. If $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ are two fuzzy soft ideals over Ω based on parameters $k \in K$ and $v \in V$, respectively, then:

$$(\tilde{\kappa}, K) \tilde{\wedge} (\tilde{\upsilon}, V) = (\tilde{\tau}, K \times V)$$

is a fuzzy soft ideal over Ω based on the parameter (k, v).

Theorem 5.12: If $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ are two fuzzy soft commutative ideals over a BCK-algebra Ω based on parameters $k \in K$ and $v \in V$, respectively, then:

$$(\tilde{\kappa}, K) \tilde{\wedge} (\tilde{\upsilon}, V) = (\tilde{\tau}, K \times V)$$

is a fuzzy soft commutative ideal over Ω based on the parameter (k, v).

Proof: Let $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ be two fuzzy soft commutative ideals over a BCK-algebra Ω based on parameters $k \in K$ and $v \in V$, respectively. Then $(\tilde{\kappa}, K)$ and $(\tilde{\upsilon}, V)$ are two fuzzy soft ideals over a BCK-algebra Ω based on parameters $k \in K$ and $v \in V$, respectively, by Theorem 5.3. Hence,

$$(\tilde{\kappa}, K) \tilde{\wedge} (\tilde{\upsilon}, V) = (\tilde{\tau}, K \times V)$$

is a fuzzy soft ideal over Ω based on the parameter (k, v), by Lemma 5.11. For any $\omega_1, \omega_2 \in \Omega$, we have:

$$\begin{aligned}\tilde{\tau}[k, v](\omega_1 * \omega_2) &= (\tilde{\kappa}[k] \cap \tilde{\upsilon}[v])(\omega_1 * \omega_2) = \min\{\tilde{\kappa}[k](\omega_1 * \omega_2), \tilde{\upsilon}[v](\omega_1 * \omega_2)\}\\&\leq \min\{\tilde{\kappa}[k](\omega_1 * (\omega_2 * (\omega_2 * \omega_1))), \tilde{\upsilon}[v](\omega_1 * (\omega_2 * (\omega_2 * \omega_1)))\}\\&= (\tilde{\kappa}[k] \cap \tilde{\upsilon}[v])(\omega_1 * (\omega_2 * (\omega_2 * \omega_1)))\\&= \tilde{\tau}[k, v](\omega_1 * (\omega_2 * (\omega_2 * \omega_1))),\end{aligned}$$

by Proposition 5.7. It follows from Theorem 5.8 that: $(\tilde{\kappa}, K) \tilde{\wedge} (\tilde{\upsilon}, V) = (\tilde{\tau}, K \times V)$

is a fuzzy soft commutative ideal over Ω based on the parameter (k, v).

Theorem 5.13: If $(\tilde{\kappa}, K)$ and $(\tilde{v}, V)$ are two fuzzy soft commutative ideals over a BCK-algebra *Ω*, then the "extended intersection" of $(\tilde{\kappa}, K)$ and $(\tilde{v}, V)$ is a fuzzy soft commutative ideal over *Ω*.

Proof: Let $(\tilde{\kappa}, K)\ \widetilde{\cap}_e\ (\tilde{v}, V) = (\tilde{\tau}, T)$ be the "extended intersection" of fuzzy soft commutative ideals $(\tilde{\kappa}, K)$ and $(\tilde{v}, V)$ over a BCK-algebra Ω. Then $T = K \cup V$. For any $t \in T$, if $t \in K\backslash V$ (resp. $t \in V\backslash K$) then $\tilde{\tau}[t] = \tilde{\kappa}[t]$ (resp. $\tilde{\tau}[t] = \tilde{v}[t]$) is a fuzzy commutative ideal. If $K \cap V \neq \emptyset$, then $\tilde{\tau}[t] = \tilde{\kappa}[t] \cap \tilde{v}[t]$ is a fuzzy commutative ideal for all $t \in K \cap V$ since the intersection of two fuzzy commutative ideals is a fuzzy commutative ideal. Therefore $(\tilde{\tau}, T)$ is a fuzzy soft commutative ideal over a BCK-algebra Ω.

The following two corollaries are straightforward results of Theorem 5.13.

Corollary 5.14: If $(\tilde{\kappa}, K)$ and $(\tilde{v}, K)$ are two fuzzy soft commutative ideals over a BCK-algebra *Ω*, then their "extended intersection" is a fuzzy soft commutative ideal over *Ω*.

Corollary 5.15: The "restricted intersection" of two fuzzy soft commutative ideals is a fuzzy soft commutative ideal.

Theorem 5.16: Let $(\tilde{\kappa}, K)$ and $(\tilde{v}, V)$ be two fuzzy soft commutative ideals over a BCK-algebra *Ω*. If K and V are disjoint, then the "union" $(\tilde{\kappa}, K)\ \widetilde{\cup}\ (\tilde{v}, V)$ is a fuzzy soft commutative ideal over *Ω*.

Proof: Since K and V are disjoint, then for all $t \in T$, either $t \in K\backslash V$ or $t \in V\backslash K$, by means of Definition 3.3. If $t \in K\backslash V$, then $\tilde{\tau}[t] = \tilde{\kappa}[t]$ is a fuzzy commutative ideal in a BCK-algebra Ω because $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over a BCK-algebra Ω. If $t \in V\backslash K$, then $\tilde{\tau}[t] = \tilde{v}[t]$ is a fuzzy commutative ideal in a BCK-algebra Ω because $(\tilde{v}, V)$ is a fuzzy soft commutative ideal over a BCK-algebra Ω. Hence $(\tilde{\tau}, T) = (\tilde{\kappa}, K)\ \widetilde{\cup}\ (\tilde{v}, V)$ is a fuzzy soft commutative ideal over a BCK-algebra Ω.

Lemma 5.17 (Meng and Jun, 1994): For a subset J of a BCK-algebra *Ω*, the following are equivalent:

1. J is a commutative ideal of Ω.
2. J is an ideal of Ω which satisfies the following condition:

$$(\forall \omega_1, \omega_2 \in \Omega)(\omega_1 * \omega_2 \in J \Rightarrow \omega_1 * (\omega_2 * (\omega_2 * \omega_1)) \in J) \tag{10}$$

Theorem 5.18: For a fuzzy soft set $(\tilde{\kappa}, K)$ over a BCK-algebra *Ω*, the following are equivalent:

1. $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over Ω based on a parameter $k \in K$.
2. $(\tilde{\kappa}, K)_k^{\geqq t}$ is a commutative ideal of Ω, for all $t \in [0,1]$ with $(\tilde{\kappa}, K)_k^{\geqq t} \neq \emptyset$.

Proof: Assume that $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over Ω based on a parameter $k \in K$. Then $(\tilde{\kappa}, K)$ is a fuzzy soft ideal over Ω based on a parameter $k \in K$, by Theorem 5.3. It follows from Theorem 4.6 that $(\tilde{\kappa}, K)_k^{\geqq t}$ is an ideal of Ω for all $t \in [0,1]$ with $(\tilde{\kappa}, K)_k^{\geqq t} \neq \emptyset$. Let $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 * \omega_2 \in (\tilde{\kappa}, K)_k^{\geqq t}$. Then $\tilde{\kappa}[k](\omega_1 * \omega_2) \geq t$, and so $\tilde{\kappa}[k](\omega_1 * (\omega_2 * (\omega_2 * \omega_1))) \geq \tilde{\kappa}[k](\omega_1 * \omega_2) \geq t$, by Proposition 5.7. Hence $\omega_1 * (\omega_2 * (\omega_2 * \omega_1)) \in (\tilde{\kappa}, K)_k^{\geqq t}$, and therefore $(\tilde{\kappa}, K)_k^{\geqq t}$ is a commutative ideal of Ω for all $t \in [0,1]$ with $(\tilde{\kappa}, K)_k^{\geqq t} \neq \emptyset$, by Lemma 5.17.

Conversely, suppose that $(\tilde{\kappa}, K)_k^{\geqq t}$ is a commutative ideal of Ω for all $t \in [0,1]$ with $(\tilde{\kappa}, K)_k^{\geqq t} \neq \emptyset$. Then $(\tilde{\kappa}, K)_k^{\geqq t}$ is an ideal of Ω for all $t \in [0,1]$ with $(\tilde{\kappa}, K)_k^{\geqq t} \neq \emptyset$, which implies, from Theorem 4.6, that $(\tilde{\kappa}, K)$ is a fuzzy soft ideal over Ω based on a parameter $k \in K$. If the condition (8) is false, then there exists $k \in K$ such that:

$$\tilde{\kappa}[k](\omega * \omega') > \tilde{\kappa}[k](\omega * (\omega' * (\omega' * \omega))) \text{ for } \omega, \omega' \in \Omega$$

Then $\tilde{\kappa}[k](\omega * \omega') \geq t_0 > \tilde{\kappa}[k](\omega * (\omega' * (\omega' * \omega)))$ for some $t_0 \in (0,1]$. It follows that $\omega * \omega' \in (\tilde{\kappa}, K)_k^{\geqq t}$ but $\omega * (\omega' * (\omega' * \omega)) \notin (\tilde{\kappa}, K)_k^{\geqq t}$. This leads to a contradiction, and hence the condition (8) is true. Therefore $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over Ω based on a parameter $k \in K$, by Theorem 5.8.

Corollary 5.19: A fuzzy soft set $(\tilde{\kappa}, K)$ over a BCK-algebra *Ω* is a fuzzy soft commutative ideal over *Ω* if and only if $(\tilde{\kappa}, K)_t^{\geqq}$ is a commutative ideal of *Ω* for all $t \in [0,1]$ with $(\tilde{\kappa}, K)_t^{\geqq} \neq \emptyset$.

Theorem 5.20: Let $(\tilde{\kappa}, K)$ and $(\tilde{\kappa}, V)$ be two fuzzy soft sets over a BCK-algebra *Ω* such that:

1. $(\tilde{\kappa}, K)\ \widetilde{\subseteq}\ (\tilde{\kappa}, V)$,
2. $(\tilde{\kappa}, K)$ is a fuzzy soft ideal over Ω.

If $(\tilde{\kappa}, V)$ is a fuzzy soft commutative ideal over Ω, then so is $(\tilde{\kappa}, K)$.

Proof: Assume that $(\tilde{\kappa}, V)$ is a fuzzy soft commutative ideal over Ω based on $k \in V$. Then $(\tilde{\kappa}, V)_k^{\geqq t}$ is a commutative ideal of Ω for all $t \in [0,1]$ with $(\tilde{\kappa}, K)_k^{\geqq t} \neq \emptyset$, by Theorem 5.18. Let $\omega_1, \omega_2 \in \Omega$ such that $\omega_1 * \omega_2 \in (\tilde{\kappa}, K)_k^{\geqq t}$. Since,

$$(\omega_1 * (\omega_1 * \omega_2)) * \omega_2 = (\omega_1 * \omega_2) * (\omega_1 * \omega_2) = 0 \in (\tilde{\kappa}, V)_k^{\geqq t},$$

it follows from (a3), (10), and assumption (1) that:

$$(\omega_1 * (\omega_2 * (\omega_2 * (\omega_1 * (\omega_1 * \omega_2))))) * (\omega_1 * \omega_2)$$
$$= (\omega_1 * (\omega_1 * \omega_2)) * (\omega_2 * (\omega_2 * (\omega_1 * (\omega_1 * \omega_2)))) \in (\tilde{\kappa}, V)_k^{\geqq t}\ \widetilde{\subseteq}\ (\tilde{\kappa}, K)_k^{\geqq t}.$$

Since $(\tilde{\kappa}, K)_k^{\gtrsim t}$ is an ideal of Ω, it follows from (1) that:

$$\omega_1 * (\omega_2 * (\omega_2 * (\omega_1 * (\omega_1 * \omega_2)))) \in (\tilde{\kappa}, K)_k^{\gtrsim t}. \quad (11)$$

Note that $\omega_1 * (\omega_1 * \omega_2) \leq \omega_1$, and so $\omega_2 * (\omega_2 * (\omega_1 * (\omega_1 * \omega_2))) \leq \omega_2 * (\omega_2 * \omega_1)$ by (a2). Thus:

$$\omega_1 * (\omega_2 * (\omega_2 * \omega_1)) \leq \omega_1 * (\omega_2 * (\omega_2 * (\omega_1 * (\omega_1 * \omega_2)))). \quad (12)$$

Using (11) and (12), we have $\omega_1 * (\omega_2 * (\omega_2 * \omega_1)) \in (\tilde{\kappa}, K)_k^{\gtrsim t}$. Hence $(\tilde{\kappa}, K)_k^{\gtrsim t}$ is a commutative ideal of Ω, by Lemma 5.17. Therefore $(\tilde{\kappa}, K)$ is a fuzzy soft commutative ideal over Ω, by Theorem 5.18.

6. Conclusion

In this paper, we applied the fuzzy soft set theory to commutative ideals of BCK-algebras. We introduced the notion of fuzzy soft commutative ideals over BCK-algebras, and investigated related properties. We provided relations between fuzzy soft ideals and fuzzy soft commutative ideals. We considered the "AND" operation, "extended intersection" and "union" of fuzzy soft (commutative) ideals. We characterized fuzzy soft (commutative) ideals. In our future study, we intend to apply the notions of the present paper to different algebras such as B-algebras, MV-algebras, subtraction algebras, MTL-algebras, EQ-algebras and lattice implication algebras, etc.

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Compliance with ethical standards

Conflict of interest

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