

A new generalization of the inverse Lomax distribution with statistical properties and applications

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ABSTRACT

In this paper, we introduce a new generalization of the inverse Lomax distribution with one extra shape parameter, the so-called power inverse Lomax (PIL) distribution, derived by using the power transformation method. We provide a more flexible density function with right-skewed, unimodal, and reversed J-shapes. The new three-parameter lifetime distribution capable of modeling decreasing, Reversed-J and upside-down hazard rates shapes. Some statistical properties of the PIL distribution are explored, such as quantile measure, moments, moment generating function, incomplete moments, residual life function, and entropy measure. The estimation of the model parameters is discussed using maximum likelihood, least squares, and weighted least squares methods. A simulation study is carried out to compare the efficiencies of different methods of estimation. This study indicated that the maximum likelihood estimates are more efficient than the corresponding least squares and weighted least squares estimates in approximately most of the situations. Also, the mean square errors for all estimates are decreasing as the sample size increases. Further, two real data applications are provided in order to examine the flexibility of the PIL model by comparing it with some known distributions. The PIL model offers a more flexible distribution for modeling lifetime data and provides better fits than other models such as inverse Lomax, inverse Weibull, and generalized inverse Weibull.

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1. Introduction

Lomax or Pareto type II distribution has been suggested by Lomax (1954) as an important model for lifetime analysis. The Lomax distribution is widely applied in some areas, such as analysis of income and wealth data, modeling business failure data, biological sciences, model firm size and queuing problems, reliability modeling, and life testing (Harris, 1968; Atkinson and Harrison, 1978; Holland et al., 2006; Corbellini et al., 2010; Hassan and Al-Ghamdi, 2009; Hassan et al., 2016). In the literature, some extensions of the Lomax distribution are available such as the Marshall–Olkin Extended-

Lomax (Ghitany et al., 2007), gamma-Lomax (Cordeiro et al., 2015), power Lomax (Rady et al., 2016), exponentiated Lomax geometric (Hassan and Abdelghafar, 2017), power Lomax Poisson (Hassan and Nassr, 2018), exponentiated Weibull-Lomax (Hassan and Abd-Allah, 2018), and Type II half logistic Lomax (Hassan et al., 2020) distributions among others.

The inverse Lomax (IL) distribution is a special case of the generalized beta distribution of the second kind. It is one of the significant lifetime models in statistical applications. Also, it has an application in various fields like stochastic modeling, economics, actuarial sciences, and life testing as discussed by Kleiber and Kotz (2003). Besides this, it has been used to obtain the Lorenz ordering relationship among ordered statistics (Kleiber, 2004). The IL distribution has been used on geophysical databases especially on the sizes of land fibers in California State of United States (McKenzie et al., 2011). The IL distribution can be derived from

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Lomax distribution using the transformation, distribution has a Lomax Z where $Y=1/Z$.

The probability density function (pdf) of two-parameter IL distribution is specified by,

$$g(y; \lambda, \beta) = \frac{\lambda}{\beta^\lambda} y^{\lambda-1} (1 + \frac{y}{\beta})^{-(\lambda+1)}; \quad \beta, \lambda, y > 0, \quad (1)$$

here, β and λ are the scale and shape parameters respectively. The associated cumulative distribution function (CDF) is:

$$G(y; \lambda, \beta) = (1 + \frac{\beta}{y})^{-\lambda}. \quad (2)$$

Rahman and Aslam (2014) used a two-component mixture IL model for the prediction of future ordered observations in the Bayesian framework using predictive models. Singh et al. (2016) obtained reliability estimates of the IL model under Type II censoring. In addition to this, the IL distribution from hybrid censored was applied to the survival data by Yadav et al. (2016).

In this paper, we provide a more flexible model by inducing just one extra shape parameter to inverse Lomax model for improving its goodness-of-fit to real data. We discuss some of its statistical properties. Estimation of the unknown parameters for the subject model using the maximum likelihood (ML), least squares (LS), and weighted least squares (WLS) methods is considered. Finally, simulation issues, as well as application to real data, are provided. The layout of the paper contains the following sections. In Section 2, we introduce the three-parameter PIL distribution. Some statistical properties of PIL distribution are presented in Section 3. In Section 4, ML, LS, and WLS estimators for the model parameters are obtained. Numerical study and analysis of real data sets are presented in Section 5. The article ends with concluding remarks.

2. Power inverse Lomax distribution

The power inverse Lomax distribution is developed by considering the power transformation $X = Y^{\frac{1}{\theta}}$, where the random variable Y follows IL distribution with parameters λ and β . The cdf of the PIL distribution is specified by,

$$F(x; \beta, \lambda, \theta) = (1 + \frac{\beta}{x^\theta})^{-\lambda}, \quad x > 0. \quad (3)$$

The pdf of PIL distribution corresponding to 3 can be written as follows:

$$f(x; \beta, \lambda, \theta) = \theta \lambda \beta x^{\lambda\theta-1} (\beta + x^\theta)^{-(\lambda+1)}. \quad \beta, \lambda, \theta, x > 0. \quad (4)$$

A random variable X has PIL distribution will be denoted by $X \sim \text{PIL}(\lambda, \beta, \theta)$. The survival function; $S(x; \lambda, \beta, \theta)$, the hazard rate function (hrf); $h(x; \lambda, \beta, \theta)$ and cumulative hazard function; $H(x; \lambda, \beta, \theta)$ of the PIL distribution are respectively given by,

$$S(x; \lambda, \beta, \theta) = 1 - (1 + \frac{\beta}{x^\theta})^{-\lambda},$$

$$h(x; \lambda, \beta, \theta) = \frac{\theta \lambda \beta x^{\lambda\theta-1} (\beta + x^\theta)^{-(\lambda+1)}}{1 - (1 + \frac{\beta}{x^\theta})^{-\lambda}},$$

and,

$$H(x; \lambda, \beta, \theta) = -\ln(1 - F(x)) = -\ln\left(1 - \left(1 + \frac{\beta}{x^\theta}\right)^{-\lambda}\right).$$

Plots of the pdf and hrf for some selected parameters values are displayed in Fig. 1 and Fig. 2. As seen from these figures that the pdf and hrf take different shapes according to different values of parameters.

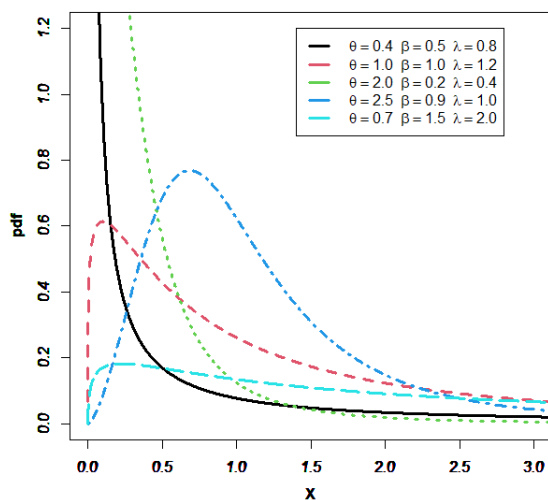


Fig. 1: The pdf of PIL distribution

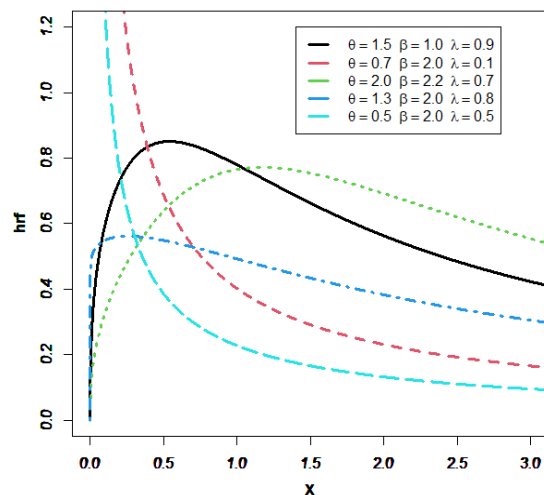


Fig. 2: The hrf of PIL distribution

3. Properties of PIL distribution

In this section, some important mathematical characteristics of the PIL distribution are developed,

specifically; the r^{th} moment, the moment-generating function, incomplete moments, moments of residual life function, and Rényi entropy.

3.1. Moments

An explicit expression of the r^{th} moment for the PIL can be obtained from pdf 4 as follows:

$$\mu_r = E(X^r) = \int_0^\infty x^r \theta \lambda \beta x^{\lambda \theta - 1} (\beta + x^\theta)^{-(\lambda+1)} dx.$$

Hence, after simplification, the r^{th} moment of PIL distribution is obtained as follows:

$$\mu_r = \frac{\beta^{\frac{r}{\theta}}}{\Gamma(\lambda)} \Gamma(1 - \frac{r}{\theta}) \Gamma(\frac{r}{\theta} + \lambda), \quad r < \theta. \tag{5}$$

Setting $r=1, 2, 3, 4$ in 5, we can obtain the first four moments about zero. Generally, the moment

generating function of PIL distribution is obtained as follows:

$$M_x(t) = E(e^{tx}) = \sum_{r=0}^\infty \frac{t^r}{r!} \beta^{\frac{r}{\theta}} \frac{\Gamma(1 - \frac{r}{\theta}) \Gamma(\frac{r}{\theta} + \lambda)}{\Gamma(\lambda)}, \quad r < \theta.$$

The r^{th} central moment (μ_r) of X is given by,

$$\mu_r = E(X - \mu_1)^r = \sum_{i=0}^r (-1)^i \frac{r!}{i!(r-i)!} (\mu_1)^i \mu_{r-i}'.$$

The mean (μ_1) and variance (σ^2) of the PIL distribution for some selected values of the parameters which can be calculated numerically in [Table 1](#). Also, the skewness (SK) and (KU)kurtosis of the PIL distribution for various values of parameters λ and θ can be calculated numerically in [Table 2](#).

Table 1: Mean and variance of PIL distribution for various values of θ, β and λ

θ	λ	$\beta = 1$		$\beta = 2$		$\beta = 3$	
		μ	σ^2	μ	σ^2	μ	σ^2
4.5	0.1	0.347	0.154	0.405	0.210	0.443	0.251
	0.2	0.544	0.190	0.635	0.259	0.695	0.310
	0.3	0.676	0.203	0.788	0.276	0.863	0.331
5	0.1	0.366	0.143	0.421	0.189	0.456	0.223
	0.2	0.563	0.167	0.646	0.220	0.701	0.259
	0.3	0.690	0.170	0.792	0.225	0.859	0.264
6	0.1	0.402	0.130	0.451	0.163	0.483	0.187
	0.2	0.597	0.135	0.670	0.170	0.716	0.195
	0.3	0.716	0.129	0.804	0.162	0.860	0.186

Table 2: Skewness and kurtosis of PIL distribution for various values of θ and λ

θ	λ	SK	KU
4.5	0.1	2.262	24.701
	0.2	2.023	27.095
	0.3	2.118	32.45
5	0.1	1.769	11.75
	0.2	1.519	12.642
	0.3	1.587	15.134
6	0.1	1.210	5.767
	0.2	0.948	6.175
	0.3	0.991	7.417

From [Table 1](#), it can be observed that both values of the mean and variance of the PIL distribution increase as the values of β and λ increase. Also, the values of the variance decrease as the values of θ increase. From [Table 2](#), it can be noticed that both the skewness and the kurtosis are decreasing functions of θ and λ .

Next, we derive a simple formula for the s^{th} incomplete moment of X defined by $\varphi_s(t) = E(X^s | X < t)$. So, the quantity $\varphi_s(t)$ comes from 4 as follows

$$\varphi_s(t) = \lambda \beta^{\frac{s}{\theta}} B\left(\frac{t^\theta}{\beta + t^\theta}; \frac{s}{\theta} + \lambda, 1 - \frac{s}{\theta}\right), \tag{6}$$

where $B\left(\frac{t^\theta}{\beta + t^\theta}; \frac{s}{\theta} + \lambda, 1 - \frac{s}{\theta}\right)$ is the incomplete beta function. The first incomplete moment of X is important to determine the mean deviations, which can be used to measure the amount of scattering in a population, and the Bonferroni and Lorenz curves. It can be obtained by substituting $s=1$ in (6). Additionally, the Bonferroni and Lorenz curves of PIL distribution are, respectively, given by,

$$B_F(t) = \frac{\Gamma(1+\lambda) B\left(\frac{t^\theta}{\beta + t^\theta}; \frac{1}{\theta} + \lambda, 1 - \frac{1}{\theta}\right)}{\Gamma(1 - \frac{1}{\theta}) \Gamma(\frac{1}{\theta} + \lambda) (1 + \frac{\beta}{t^\theta})^{-\lambda}},$$

and,

$$L_F(t) = \frac{\Gamma(1+\lambda)}{\beta^{\frac{1}{\theta}} \Gamma(1 - \frac{1}{\theta}) \Gamma(\frac{1}{\theta} + \lambda)} \beta^{\frac{1}{\theta}} B\left(\frac{t^\theta}{\beta + t^\theta}; \frac{1}{\theta} + \lambda, 1 - \frac{1}{\theta}\right).$$

3.2. Moments of residual life function

The n^{th} moment of the residual life of X is given by,

$$m_n(t) = \frac{1}{s(t)} \int_t^\infty (x - t)^n f(x) dx.$$

Hence, n^{th} moment of residual life of PIL distribution can be obtained as follows:

$$m_n(t) = \frac{1}{s(t)} \int_t^\infty (x - t)^n f(x) dx = \frac{1}{s(t)} \sum_{r=0}^n \frac{n!}{r!(n-r)!} (-1)^{n-r} t^{n-r} \lambda \beta^{\frac{r}{\theta}} B\left(\frac{\beta}{\beta + t^\theta}; 1 - \frac{r}{\theta}, \frac{r}{\theta} + \lambda\right).$$

where $B\left(\frac{\beta}{\beta + t^\theta}; 1 - \frac{r}{\theta}, \frac{r}{\theta} + \lambda\right)$ is the incomplete beta function.

3.3. Rényi entropy

Entropy is a measure of uncertainty of a random variable X . The Rényi entropy of random variable X for a continuous random variable with range R is defined as follows:

$$I_R(X) = \frac{1}{1-\delta} \ln \left\{ \int f(x)^\delta dx \right\}, \quad \delta \neq 1, \delta > 0.$$

The pdf; $(f(x; \lambda, \beta, \theta))^\delta$ of the PIL distribution can be expressed as follows:

$$f(x)^\delta = (\theta\lambda\beta)^\delta x^{\delta(\lambda\theta-1)}(\beta + x^\theta)^{-\delta(\lambda+1)}.$$

Therefore, the Rényi entropy of PIL distribution is given by,

$$I_R(X) = \frac{1}{1-\delta} \ln \left\{ \lambda^\delta \theta^{\delta-1} \beta^{\frac{1-\delta}{\theta}} \frac{\Gamma(\frac{\delta}{\theta} + \delta) \Gamma(\frac{1-\delta}{\theta} + \lambda\delta)}{\Gamma(\delta + \lambda\delta)} \right\}.$$

Table 3 gives $I_R(X)$ of the PIL distribution for different choices of parameters θ , β and λ . It's seems that the entropy increases with increasing values of λ and β , while decreases with increasing values of θ .

Table 3: Entropy for several arbitrary parameter values

$\lambda = 1, \beta = 2$		$\theta = 1, \beta = 3$		$\theta = 2, \lambda = 0.5$	
θ	Entropy	λ	Entropy	β	Entropy
1.0	1.792	0.8	1.766	0.5	0.183
1.5	1.553	1.5	2.773	1.5	0.732
2.0	1.281	2.0	3.114	2.0	0.876
2.5	1.048	2.7	3.443	2.5	0.987
3.0	0.852	3.5	3.717	3.0	1.079
4.0	0.538	5.0	4.084	4.5	1.281

4. Parameter estimation

This section deals with the parameter estimation for PIL distribution based on ML, LS, and WLS methods.

4.1. Maximum likelihood estimator

Let $X_1, X_2, X_3, \dots, X_n$ be the observed values follow PIL distribution with pdf 4, then the log-likelihood function, denoted by lnL , based on a complete sample for the unknown parameters can be expressed as:

$$lnL = nln\theta + nln\lambda + nln\beta + (\lambda\theta - 1) \sum_{i=1}^n \ln(x_i) - (\lambda + 1) \sum_{i=1}^n \ln(\beta + x_i^\theta).$$

The partial derivatives of the log-likelihood function with respect to θ , λ and β can be obtained as follows:

$$\frac{\partial lnL}{\partial \theta} = \frac{n}{\theta} + \lambda \sum_{i=1}^n \ln(x_i) - (\lambda + 1) \sum_{i=1}^n \frac{x_i^\theta \ln(x_i)}{\beta + x_i^\theta},$$

$$\frac{\partial lnL}{\partial \lambda} = \frac{n}{\lambda} + \theta \sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n \ln(\beta + x_i^\theta),$$

and,

$$\frac{\partial lnL}{\partial \beta} = \frac{n}{\beta} - (\lambda + 1) \sum_{i=1}^n \frac{1}{\beta + x_i^\theta}.$$

Then ML estimators of the parameters θ , λ , and β denoted by $\hat{\theta}$, $\hat{\lambda}$ and $\hat{\beta}$ are determined by solving numerically the non-linear equations $\partial lnL / \partial \theta = 0$, $\partial lnL / \partial \lambda = 0$, and $\partial lnL / \partial \beta = 0$ simultaneously.

4.2. Least squares estimator

The LS estimators were originally proposed by Swain et al. (1988) to estimate the parameters of the

beta distribution. The method of LS is about estimating parameters by minimizing the sum of square errors between the observed data and their expected values with respect to the unknown parameters. That is;

$$\sum_{i=1}^n [[F(x_{i:n})] - E(F(x_{i:n}))]^2,$$

also, the WLS can be obtained by minimizing the following with respect to the unknown parameters:

$$\sum_{i=1}^n w_i [[F(x_{i:n})] - E(F(x_{i:n}))]^2,$$

where,

$$w_i = \frac{1}{var(F(x_{i:n}))} = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

Let $X_1, X_2, X_3, \dots, X_n$ be a random sample of size n from PIL distribution. Suppose that $X_{1:n} < X_{2:n} < X_{3:n} < \dots < X_{n:n}$ denotes the corresponding ordered sample. Therefore, the LS estimators of θ , λ and β say, $\bar{\theta}$, $\bar{\lambda}$ and $\bar{\beta}$ respectively, can be obtained by minimizing the following function with respect to θ , λ , and β ,

$$\sum_{i=1}^n [(1 + \frac{\beta}{x_{i:n}})^{-\lambda} - \frac{i}{n+1}]^2.$$

Also, the WLS estimators of θ , λ and β say, $\bar{\theta}$, $\bar{\lambda}$ and $\bar{\beta}$ respectively, can be obtained by minimizing the following quantity with respect to θ , λ , and β ,

$$\sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} [(1 + \frac{\beta}{x_{i:n}})^{-\lambda} - \frac{i}{n+1}]^2.$$

5. Numerical illustration

In this section, a numerical study is presented to compare the performance of the estimates for different parameter values. The performance of the estimates of unknown parameters has been measured in terms of their *absolute bias* (AB), *standard error* (SE), and *mean square error* (MSE) for different sample sizes and for different parameter values. The numerical procedures are defined through the following algorithm.

Step 1: A random sample x_1, \dots, x_n of size $n = 10, 20, 30, 50$ are selected, these random samples are generated from the PIL distribution by using the transformation $x_i = \beta^{\frac{1}{\theta}} [u_i^{\frac{-1}{\lambda}} - 1]^{\frac{-1}{\theta}}$, $i=1, 2, \dots, n$ and u_i are random samples from uniform (0,1).

Step 2: Six different set of the parameters are selected as, Set1=($\lambda = 1, \theta = 1.2, \beta = 1$) Set2=($\lambda = 0.8, \theta = 1.2, \beta = 1$), Set3=($\lambda = 1.4, \theta = 1, \beta = 0.7$) Set4=($\lambda = 1.4, \theta = 0.8, \beta = 0.7$) Set5=($\lambda = 1, \theta = 1.2, \beta = 1.5$), Set6=($\lambda = 1.4, \theta = 1.2, \beta = 1.5$)

Step 3: For each set of parameters and for each sample size, the ML, LS, and WLS estimates of λ , θ and β are calculated.

Step 4: Steps from 1 to 3 are repeated 1000 times for each sample size and for selected sets of

parameters. Then, the ABs, SEs, and MSEs of the estimates of the unknown parameters are calculated.

5.1. Numerical results

Numerical results are reported in Tables 4 to 6 and represented through some Figs. 3-6. From these tables and figures, the following observations can be

detected on the properties of estimated parameters from the PIL distribution.

1. The MSEs and SEs of the ML, LS, and WLS' estimates decrease as the sample sizes increase for a different selected set of parameters (Fig. 3 and Fig. 4).

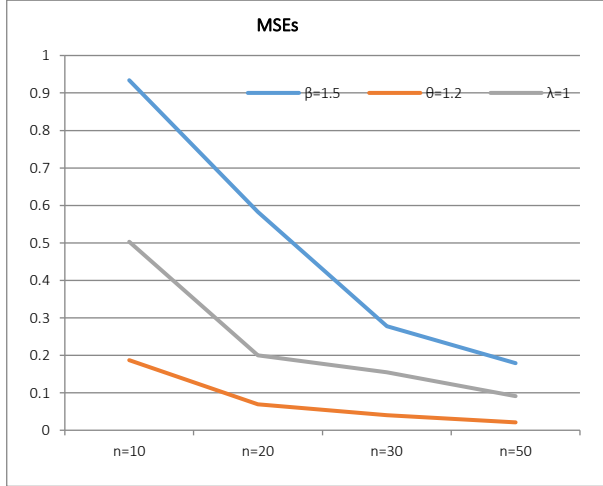


Fig. 3: MSE for ML estimates of PIL distribution

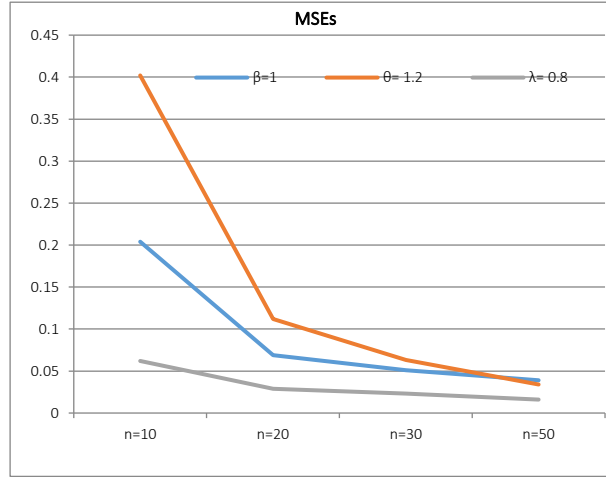


Fig. 4: MSE for LS estimates of PIL distribution

2. The MSEs for the WLS estimates $\bar{\lambda}$ take the smallest value among the corresponding MSEs for the other methods in almost all of the cases (Tables 4-6).

3. As it seems from Fig. 5, the MSEs of the ML estimates of λ take the largest values corresponding to the LS and WLS estimates for the same sample size. Also, the MSEs of ML estimates of λ for all sets of parameters have the largest values for the same

sample size. The Set 2 of parameters gives the smallest MSE for different λ estimates corresponding to other sets of parameters.

4. As it seems from Fig. 6, the SEs of the ML estimates of θ take the smallest values corresponding to the LS and WLS estimates for the same sample size in most cases. The Set 4 of parameters gives the smallest SE for different θ estimates corresponding to other sets of parameters.

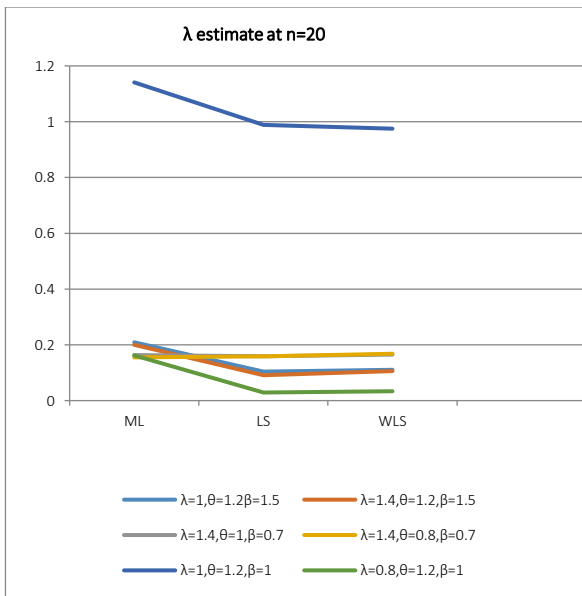


Fig. 5: MSEs of $\hat{\lambda}$, $\tilde{\lambda}$ and $\bar{\lambda}$ for all sets of parameters

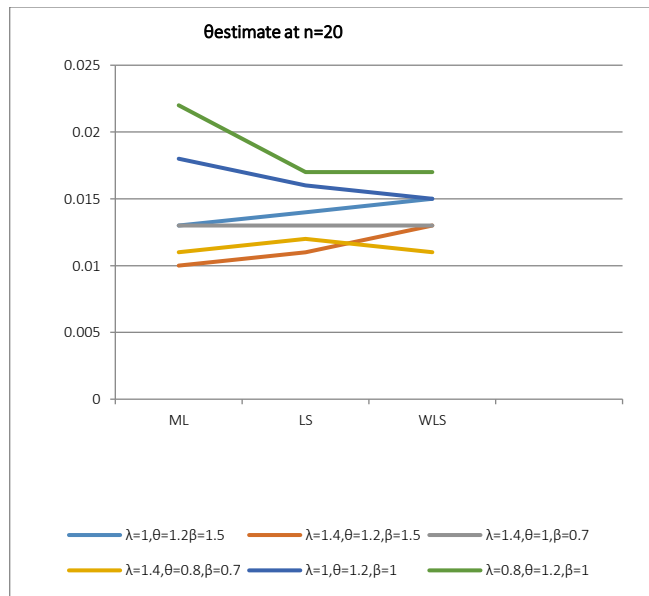


Fig. 6: SEs of $\hat{\theta}$, $\tilde{\theta}$ and $\bar{\theta}$ for all sets of parameters

5. For a fixed value of θ , β the ABs of $\hat{\lambda}$, $\tilde{\lambda}$ and $\bar{\lambda}$ in all methods increase as the value of λ decreases (Tables 4 and 6). For fixed values of λ , β and as the values of θ increase, the ABs, SEs, and MSEs for the ML estimates are decreasing, (Table 4). As the values of

λ increase and for fixed values of θ and β , the SEs and MSEs for LS, WLS estimates increase (Table 6).

6. The SEs for the LS estimates, $\tilde{\lambda}$ take the smallest value among the corresponding SEs for the other methods in almost all of the cases.

Table 4: ABs, MSEs, and SEs of estimates for set1 and set 2 of PIL distribution

n	Method	Properties	$\lambda = 1, \theta = 1.2, \beta = 1$			$\lambda = 0.8, \theta = 1.2, \beta = 1$		
			λ	θ	β	λ	θ	β
10	ML	MSE	1.19700	1.63700	1.19900	0.43200	1.54600	1.37900
		AB	0.1970	0.43700	0.19900	0.25700	0.35200	0.11300
		SE	0.05900	0.31000	0.10700	0.06100	0.11900	0.11700
	LS	MSE	0.99400	1.25000	1.06700	0.06200	0.40200	0.20400
		AB	0.00626	0.05000	0.06700	0.08100	0.03800	0.04500
		SE	0.02500	0.05600	0.03700	0.02400	0.06300	0.04500
	WLS	MSE	0.98300	1.27100	1.10200	0.06200	0.44100	0.16000
		AB	0.01700	0.07100	0.10200	0.07100	0.06200	0.03000
		SE	0.02500	0.06400	0.04200	0.02400	0.06600	0.04000
20	ML	MSE	1.14100	1.29700	1.02100	0.16200	0.20200	0.59000
		AB	0.14100	0.09700	0.02100	0.14400	0.09900	0.01700
		SE	0.02100	0.01800	0.03400	0.01900	0.02200	0.03800
	LS	MSE	0.98800	1.21200	1.04800	0.02900	0.11200	0.06900
		AB	0.01200	0.01200	0.04800	0.06100	0.02700	0.09000
		SE	0.00823	0.01600	0.01400	0.00789	0.01700	0.01200
	WLS	MSE	0.97500	1.22600	1.08600	0.03300	0.11100	0.10300
		AB	0.02500	0.02600	0.08600	0.04100	0.00126	0.04100
		SE	0.00839	0.01500	0.01700	0.00888	0.01700	0.01600
30	ML	MSE	0.10300	0.06900	0.31600	0.10300	0.07900	0.25600
		AB	0.10400	0.06300	0.02300	0.14100	0.04400	0.10500
		SE	0.01000	0.00852	0.01600	0.00960	0.00925	0.01700
	LS	MSE	0.01800	0.05500	0.06000	0.02300	0.06300	0.05100
		AB	0.00430	0.00781	0.05200	0.07000	0.02000	0.09300
		SE	0.00440	0.00826	0.00817	0.00444	0.00834	0.00686
	WLS	MSE	0.02000	0.04900	0.09700	0.02600	0.06700	0.07300
		AB	0.02700	0.02900	0.10500	0.04700	0.00871	0.04300
		SE	0.00488	0.00860	0.00977	0.00524	0.00866	0.00902
50	ML	MSE	0.05000	0.03800	0.13300	0.05300	0.03800	0.11900
		AB	0.07300	0.02600	0.03100	0.11600	0.01300	0.12000
		SE	0.00457	0.00385	0.00726	0.00399	0.00390	0.00646
	LS	MSE	0.01200	0.03200	0.03800	0.01600	0.03400	0.03900
		AB	0.00865	0.00945	0.03300	0.04700	0.04300	0.08800
		SE	0.00219	0.00357	0.00385	0.00207	0.00357	0.00354
	WLS	MSE	0.01600	0.03600	0.06100	0.01900	0.03000	0.07700
		AB	0.03400	0.03100	0.08300	0.04400	0.01700	0.02200
		SE	0.00246	0.00375	0.00475	0.00264	0.00348	0.00568

Table 5: ABs, MSEs, and SEs of estimates for set3 and set 4 of PIL distribution

N	Method	Properties	$\lambda = 1.4, \theta = 1, \beta = 0.7$			$\lambda = 1, \theta = 0.8, \beta = 0.7$		
			λ	θ	β	λ	θ	β
10	ML	MSE	0.38100	0.61700	1.19300	0.29000	0.20000	0.89700
		AB	0.06800	0.31700	0.38500	0.09800	0.22300	0.36400
		SE	0.06100	0.07200	0.10200	0.05300	0.03900	0.08700
	LS	MSE	0.20500	0.31700	0.28400	0.20200	0.36800	0.37300
		AB	0.35100	0.12700	0.39000	0.34000	0.10600	0.42900
		SE	0.02900	0.05500	0.03600	0.02900	0.06000	0.04300
	WLS	MSE	0.20900	1.14700	0.35300	0.23100	0.89200	0.44800
		AB	0.35000	0.18700	0.42000	0.34700	0.16700	0.46900
		SE	0.02900	0.10600	0.04200	0.03300	0.09300	0.04800
20	ML	MSE	0.16300	0.08600	0.26200	0.15500	0.06600	0.31400
		AB	0.12200	0.12900	0.23000	0.16200	0.11500	0.26700
		SE	0.01900	0.01300	0.02300	0.01800	0.01100	0.02500
	LS	MSE	0.15900	0.07600	0.24000	0.15900	0.06000	0.29500
		AB	0.36200	0.08500	0.38700	0.35900	0.07000	0.41400
		SE	0.00833	0.01300	0.01500	0.00866	0.01200	0.01800
	WLS	MSE	0.16500	0.07400	0.34600	0.16800	0.05800	0.34000
		AB	0.36600	0.09900	0.42900	0.36600	0.08500	0.44400
		SE	0.00883	0.01300	0.02000	0.00920	0.01100	0.01900
30	ML	MSE	0.13000	0.04800	0.23600	0.10200	0.03000	0.16600
		AB	0.15300	0.09800	0.23700	0.16800	0.07800	0.21600
		SE	0.01100	0.00654	0.01400	0.00904	0.00516	0.01200
	LS	MSE	0.14500	0.05100	0.23800	0.14700	0.03400	0.27600
		AB	0.35600	0.08100	0.39700	0.35400	0.06400	0.39400
		SE	0.00453	0.00702	0.00946	0.00486	0.00579	0.01200
	WLS	MSE	0.15200	0.05700	0.28300	0.15200	0.03300	0.28200
		AB	0.35800	0.09900	0.42300	0.35400	0.07800	0.41600
		SE	0.00510	0.00729	0.01100	0.00542	0.00549	0.01100
50	ML	MSE	0.08400	0.03000	0.12300	0.08000	0.01700	0.09800
		AB	0.18100	0.08400	0.21000	0.18900	0.06300	0.20300
		SE	0.00453	0.00304	0.00562	0.00422	0.00230	0.00474
	LS	MSE	0.13100	0.03400	0.16800	0.13500	0.02100	0.21700
		AB	0.34600	0.08600	0.36000	0.34900	0.06800	0.37800
		SE	0.00213	0.00323	0.00394	0.00234	0.00259	0.00546
	WLS	MSE	0.14000	0.03400	0.21400	0.13600	0.02100	0.22100
		AB	0.35400	0.10300	0.39600	0.34600	0.07700	0.38900
		SE	0.00245	0.00311	0.00482	0.00256	0.00246	0.00529

Table 6: ABs, MSEs, and SEs of estimates for set5 and set 6 of PIL distribution

n	Method	Properties	$\lambda = 1, \theta = 1.2, \beta = 1.5$			$\lambda = 1.4, \theta = 1.2, \beta = 1.5$		
			λ	θ	β	λ	θ	β
10	ML	MSE	0.50300	0.18700	0.93400	0.49900	0.16500	0.79900
		AB	0.31500	0.14200	0.00809	0.31600	0.14600	0.04000
		SE	0.06400	0.04100	0.09700	0.06300	0.03800	0.08900
	LS	MSE	0.23400	0.15300	0.20500	0.35900	0.32500	0.33900
		AB	0.07600	0.03300	0.03500	0.10400	0.01500	0.00680
		SE	0.04800	0.03900	0.04500	0.05900	0.05700	0.05800
	WLS	MSE	0.22700	0.14200	0.31800	0.39300	0.52300	0.47100
		AB	0.03400	0.01300	0.07300	0.05600	0.03400	0.11500
		SE	0.04800	0.03800	0.05700	0.06300	0.07200	0.06800
20	ML	MSE	0.20000	0.06900	0.58300	0.20900	0.04500	0.48500
		AB	0.18500	0.07300	0.00862	0.18900	0.04500	0.04000
		SE	0.02000	0.01300	0.03800	0.02100	0.01000	0.03500
	LS	MSE	0.09100	0.07400	0.10200	0.10400	0.05000	0.14300
		AB	0.06300	0.00359	0.02400	0.02500	0.01200	0.04000
		SE	0.01500	0.01400	0.01600	0.01600	0.01100	0.01900
	WLS	MSE	0.10600	0.09400	0.32200	0.11000	0.06700	0.44700
		AB	0.04300	0.04000	0.22700	0.07000	0.02400	0.28300
		SE	0.01700	0.01500	0.02900	0.01700	0.01300	0.03300
30	ML	MSE	0.15500	0.04000	0.27800	0.13000	0.02700	0.27600
		AB	0.17500	0.03800	0.09600	0.15200	0.02900	0.07900
		SE	0.01200	0.00652	0.01700	0.01100	0.00539	0.01700
	LS	MSE	0.06700	0.04600	0.05900	0.07000	0.02900	0.12100
		AB	0.05500	0.00788	0.03700	0.02200	0.00174	0.03500
		SE	0.00842	0.00717	0.00802	0.00879	0.00571	0.01200
	WLS	MSE	0.06700	0.05800	0.30500	0.06000	0.03100	0.48300
		AB	0.08100	0.04100	0.27200	0.10400	0.03400	0.34000
		SE	0.00935	0.00806	0.01900	0.00851	0.00591	0.02300
50	ML	MSE	0.09100	0.02100	0.17900	0.09600	0.01800	0.21100
		AB	0.14000	0.00794	0.12600	0.13800	0.01100	0.10300
		SE	0.00536	0.00287	0.00808	0.00555	0.00268	0.00896
	LS	MSE	0.04000	0.02200	0.04200	0.04700	0.01900	0.10500
		AB	0.02900	0.01800	0.03000	0.00717	0.00323	0.06400
		SE	0.00397	0.00293	0.00404	0.00434	0.00275	0.00635
	WLS	MSE	0.06100	0.03000	0.42500	0.08500	0.03300	0.49200
		AB	0.12500	0.03900	0.35500	0.11600	0.04100	0.36600
		SE	0.00524	0.00357	0.00130	0.00579	0.00363	0.01300

5.2. Data analysis

In this subsection, two real data sets are provided to illustrate the importance of the PIL distribution by comparing it with some other distributions (IL, inverse Weibull (IW), and generalized inverse Weibull (GIW)). Two real data sets are used to show that PIL distribution can be applied in practice and can be a better model than some others. The first data set is corresponding to remission times (in months) of a random sample of 128 bladder cancer patients given in [Lee and Wang \(2003\)](#). The data are given as follows:

0.08 2.09 3.48 4.87 6.94 8.66 13.11 23.63 0.2 2.23
 0.52 4.98 6.97 9.02 13.29 0.4 2.26 3.57 5.06 7.09
 0.22 13.8 25.74 0.5 2.46 3.46 5.09 7.26 9.47 14.24
 0.82 0.51 2.54 3.7 5.17 7.28 9.74 14.76 26.31 0.81
 0.62 3.28 5.32 7.32 10.06 14.77 32.15 2.64 3.88 5.32
 0.39 10.34 14.38 34.26 0.9 2.69 4.18 5.34 7.59 10.66
 0.96 36.66 1.05 2.69 4.23 5.41 7.62 10.75 16.62

43.01 0.19 2.75 4.26 5.41 7.63 17.12 46.12 1.26 2.83
 4.33 0.66 11.25 17.14 79.05 1.35 2.87 5.62 7.87
 11.64 17.36 0.4 3.02 4.34 5.71 7.93 11.79 18.1 1.46
 4.4 5.85 0.26 11.98 19.13 1.76 3.25 4.5 6.25 8.37
 12.02 2.02 0.31 4.51 6.54 8.53 12.03 20.28 2.02 3.36
 6.76 12.07 0.73 2.07 3.36 6.39 8.65 12.63 22.69 5.49.

The maximum likelihood method is performed to obtain the point estimates of the model parameters. To compare the fitted models, some selected measures are applied. The selected measures include; -2log-likelihood function evaluated at the parameter estimates, Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC). The better model is corresponding to the lower values of, AIC, CAIC, BIC, and HQIC. The results for the previous measures to the mentioned models are listed in [Table 7](#).

Table 7: The statistics -2logL, AIC, CAIC, BIC and HQIC for first data set

Distribution	-2logL	AIC	BIC	CAIC	HQIC
PIL	800.426	806.426	814.982	806.619	809.902
IL	824.528	830.528	839.084	830.721	834.004
GIW	874.450	880.450	863.673	880.644	883.926
IW	857.352	861.352	861.566	861.448	863.669

The results, in [Table 7](#), show that the PIL distribution has the smallest values of AIC, CAIC, BIC, and HQIC. Then PIL distribution provides a

significantly better fit than other distributions considered here IL, IW, and GIW. The second data set refers to [Murthy et al. \(2004\)](#) about the time

between failures for the repairable item. The data are listed as the following:

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

The results in Table 8 indicate that the PIL model is suitable for this data set based on the selected criteria. The PIL model has the lowest AIC, BIC, CAIC, and HQIC values. Therefore, the PIL is a preferable model to the other model for this data set.

Table 8: The statistics $-2\log L$, AIC, CAIC, BIC, and HQIC for the second data set

Distribution	$-2\log L$	AIC	BIC	CAIC	HQIC
PIL	79.774	85.774	89.978	86.697	87.119
IL	92.027	98.027	102.23	98.95	99.372
GIW	92.751	98.751	102.955	99.674	100.096
IW	92.751	96.751	99.554	97.196	97.648

6. Concluding remarks

In this paper, we introduce a new model, the so-called, power inverse Lomax distribution. Several properties of the PIL distribution are investigated, including the moments, incomplete moments, moments of residual life, and Rényi entropy. The estimation of population parameters is discussed through the method of the maximum likelihood, least squares, and weighted least squares. The simulation study is presented to compare the performance of estimates. Applications of the power inverse Lomax distribution to real data show that the new distribution can be used quite effectively to provide better fits than the inverse Lomax, inverse Weibull, and generalized inverse Weibull models. The power inverse Lomax distribution parameters can be investigated using ranked set sampling methods (Al-Saleh and Al-Omari, 2002; Al-Omari, 2010; 2011; Haq et al., 2014a; 2014b).

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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