International Journal of Advanced and Applied Sciences, 8(3) 2021, Pages: 57-62

Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

Comparison of classical and Bayesian estimators to estimate the parameters in Weibull distribution under weighted general entropy loss function

Fuad Alduais [1, 2, *]

[1]*Mathematics Department, College of Humanities and Science in Al Aflaj, Prince Sattam Bin Abdulaziz University, Al-Kharj, Saudi Arabia*
[2]*Business Administration Department, Administrative Science College, Thamar University, Thamar, Yemen*

ARTICLE INFO

Article history:
Received 29 August 2020
Received in revised form
19 November 2020
Accepted 21 November 2020

Keywords:
Weighted general entropy
Bayesian estimation
Weibull distribution
Classical estimation

ABSTRACT

In this work, we have developed a General Entropy loss function (GE) to estimate parameters of Weibull distribution (WD) based on complete data when both shape and scale parameters are unknown. The development is done by merging weight into GE to produce a new loss function called the weighted General Entropy loss function (WGE). Then, we utilized WGE to derive the parameters of the WD. After, we compared the performance of the developed estimation in this work with the Bayesian estimator using the GE loss function. Bayesian estimator using square error (SE) loss function, Ordinary Least Squares Method (OLS), Weighted Least Squared Method (WLS), and maximum likelihood estimation (MLE). Based on the Monte Carlo simulation method, those estimators are compared depending on the mean squared errors (MSE's). The results show that the performance of the Bayes estimator under developed method (WGE) loss function is the best for estimating shape parameters in all cases and has good performance for estimating scale parameter.

1. Introduction

The Weibull distribution (WD) is one of the most popular and widely used. Distributions in life testing and reliability studies. The Weibull distribution is mainly used in forecasting failure rates in numerous applications in various areas; for example, survival analysis, animal bioassay, breaking strength, and life expectancy.

Many researchers estimated the parameters of WD using several methods, including Bayesian and non-Bayesian (Aslam et al., 2014; Al-Duais and Alhagyan, 2020; Pobočíková and Sedliačková, 2014; Gupta and Singh, 2017; Guure et al., 2014; Mohammed and Ibrahim, 2011; Basumatary et al., 2005; Nwobi and Ugomma, 2014; Marks, 2005; Saeed et al., 2019; Al-Duais, 2020).

In this paper, we will derive a Bayesian estimator under weighted General Entropy loss function (WGE) to estimate parameters of the Weibull distribution based on complete data. After, we will compare the proposed estimator with others.

The probability density function (p. d. f) of two parameters WD which given by,

$$f(x\,;\ \theta,\eta) = \frac{\eta}{\theta} x^{\eta-1} \exp\left[-\frac{x^{\eta}}{\theta}\right] \qquad ;x>0 \quad \theta,\eta>0 \tag{1}$$

The cumulative distribution function (c. d. f) is:

$$F(x\,;\ \theta,\eta) = 1 - \exp\left[-\frac{x^{\eta}}{\theta}\right] \qquad ;x>0 \quad \theta,\eta>0 \tag{2}$$

The reliability function at time t is given by

$$R(t\,;\ \theta,\eta) = \exp\left[-\frac{t^{\eta}}{\theta}\right] \qquad ;x>0 \quad \theta,\eta>0 \tag{3}$$

where, η and θ are shape and scale parameters, respectively.

2. Classical methods of estimation of Weibull parameters

The Classical methods selected for the comparative study are (i) Maximum Likelihood Estimator (MLE), (ii) Ordinary Least Squares Method (OLS), (iii) Weighted Least Squared Method (WLS).

* Corresponding Author.
Email Address: f.alduais@psau.edu.sa
https://doi.org/10.21833/ijaas.2021.03.008
Corresponding author's ORCID profile:
https://orcid.org/0000-0002-6798-6295
2313-626X/© 2021 The Authors. Published by IASE.

2.1. Maximum likelihood estimator (MLE)

Let $\underline{x} = x_1, x_2, x_3, \ldots x_n$ be the lifetime of a random sample of size n drawn independently from the WD defined by 1, then the likelihood function for the given sample observations:

$$L\, f(\underline{x}\,;\ \theta,\eta) = \prod_{i=1}^{n} \frac{\theta}{\eta} x_i^{\eta-1} \exp\left[-\frac{x_i^{\eta}}{\theta}\right] = \frac{\eta^n}{\theta^n} \prod_{i=1}^{n} x_i^{\eta-1} \exp\left[-\frac{1}{\theta}\sum_{i=1}^{n} x_i^{\eta}\right] \quad (4)$$

Taking the natural logarithm of both sides yields,

$$\ln L = n \ln \eta - n \ln \theta + (\eta - 1) \sum_{i=1}^{n} \ln x_i - \frac{1}{\theta} \sum_{i=1}^{n} x_i^{\eta} \quad (5)$$

Differentiating (5) with respect to η and θ in turn and equating to zero, we obtain the equations as follows:

$$\frac{\partial \ln L}{\partial \eta} = \frac{n}{\hat{\eta}} - \frac{\sum_{i=1}^{n} x_i^{\hat{\eta}} \ln x_i}{\hat{\theta}} + \sum_{i=1}^{n} \ln x_i = 0\,, \quad (6)$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\hat{\theta}} + \frac{\sum_{i=1}^{n} x_i^{\hat{\eta}}}{\hat{\theta}^2} = 0 \quad (7)$$

Unfortunately, there is no closed solution for Eqs. 6 and 7. Thus, as an alternative, numerical techniques are used like the Newton-Raphson method and the iteration method.

Eqs. 6 and 7 are nonlinear, so we will use the numerical analysis by the Newton–Raphson method to find the estimate of the parameters η and θ.

2.2. Ordinary least squares method (OLS)

Let $x_1, x_2, x_3, \ldots x_n$ be a random sample of size n from the WD of Eq. 1. Form Eq. 2, its distribution given by,

$$\ln[-\ln(1 - F(x;\ \theta,\eta)] = -\ln\theta + \eta \ln x. \quad (8)$$

Let $x_{(1)} < x_{(2)} < x_{(3)} < \cdots < x_{(n)}$ be the ordered observations in a random sample of size n. Then Eq. 8 can be rewritten as:

$$\ln\left[-\ln\left(1 - F(x_{(i)};\ \theta,\eta)\right)\right] = -\ln\theta + \eta \ln x_{(i)} \quad i = 1,2,3,\ldots,n. \quad (9)$$

Eq. 9 represents a simple linear regression function corresponding to $F(x_i;\ \theta,\eta)$:

$$Y_i = a + bX_i + \epsilon_i$$

where, $Y_i = \ln\left[-ln\left(1 - \hat{F}_i\right)\right]$ and $\hat{F}_i$ it is a point estimator of $F(x_{(i)};\ \theta,\eta)$, many estimators for $\hat{F}_i$ are used, for example, the mean rank estimator $\hat{F}_i = i/n + 1$, the Median Rank estimator $\hat{F}_i = (i - 0.3)/(n + 0.4)$, $\hat{F}_i = (i - 3/8)/(n + 0.25)$ and $\hat{F}_i = (i - 0.05)/n$. $X_i = \ln x_{(i)}$, $a = -\ln\theta$, $b = \eta$.

The estimates $\hat{a}$ and $\hat{b}$ of the regression parameters, a and b minimize the function,

$$Q(a\,,b\,) = \sum_{i=1}^{n} (Y_i - a - b \ln x_{(i)}\,)^2. \quad (10)$$

Therefore, the estimates $\hat{a}$ and $\hat{b}$ of the parameters, a and b are given by,

$$\hat{b}_{OLS} = \frac{n\sum_{i=1}^{n} \ln x_{(i)}\ \ln[-\ln(1-\hat{F}_i)] - \sum_{i=1}^{n} \ln x_i\ \sum_{i=1}^{n} \ln[-\ln(1-\hat{F}_i)]}{n\ \sum_{i=1}^{n} \ln^2(x_i) - \left(\sum_{i=1}^{n} \ln x_{(i)}\right)^2} \quad (11)$$

$$\hat{a}_{OLS} = \frac{1}{n}\sum_{i=1}^{n} \ln\left[-\ln\left(1 - \hat{F}_i\right)\right] - \hat{b}_{OLS}\, \frac{1}{n}\sum_{i=1}^{n} \ln x_{(i)} \quad (12)$$

The estimates $\hat{\eta}_{OLS}$ and $\hat{\theta}_{OLS}$ of the parameters, η and θ are given by,

$$\hat{\eta}_{OLS} = \frac{n\sum_{i=1}^{n} \ln x_i\ \ln[-\ln(1-\hat{F}_i)] - \sum_{i=1}^{n} \ln x_i\ \sum_{i=1}^{n} \ln[-\ln(1-\hat{F}_i)]}{n\ \sum_{i=1}^{n} \ln^2(x_i) - \left(\sum_{i=1}^{n} \ln x_{(i)}\right)^2} \quad (13)$$

$$\hat{\theta}_{OLS} = exp\left[-\frac{1}{n}\sum_{i=1}^{n} \ln\left[-\ln\left(1 - \hat{F}_i\right)\right] - \hat{b}_{OLS}\, \frac{1}{n}\sum_{i=1}^{n} \ln x_{(i)}\right] \quad (14)$$

2.3. Weighted least squared method (WLS)

The weighted least–squares (WLS) estimate of the parameters, η, and θ are the values of the parameters which minimizes the function,

$$Q^*_{\mathcal{W}}(\eta\,,\theta\,) = \sum_{i=1}^{n} \mathcal{W}_i (\ln[-\ln\ (1 - \hat{F}_i)] + \ln\theta - \eta \ln x_{(i)}\,)^2 \quad (15)$$

The major difficulty in applying the WLS method is in finding $\mathcal{W}_i$ in Eq. 14. To obtain the weights $\mathcal{W}_i$ (Hung and Liu, 2004) used the delta method and obtained,

$$Var\left(\ln[-\ln\ (1 - \hat{F}_i)]\,\right) \propto \frac{1}{[(1-\hat{F}_i)\ln(1-\hat{F}_i)]^2} \quad (16)$$

Hence, the weights can be taken to be as follows:

$$\mathcal{W}_i = \frac{[(1-\hat{F}_i)\ln(1-\hat{F}_i)]^2}{\sum_{i=1}^{n}[(1-\hat{F}_i)\ln(1-\hat{F}_i)]^2}, \quad i = 1\,,2,3,\ldots,n. \quad (17)$$

Minimizing $Q^*_{\mathcal{W}}(\eta\,,\theta\,)$ we obtain the WLS estimates of η, and θ are:

$$\hat{\eta}_{WOLS} = \frac{\sum_{i=1}^{n} \mathcal{W}_i\, Y_i \psi_i - (\sum_{i=1}^{n} \mathcal{W}_i\, Y_i)(\sum_{i=1}^{n} \mathcal{W}_i\, \psi_i)}{\sum_{i=1}^{n} \mathcal{W}_i\, Y_i^{\,2} - \left(\sum_{i=1}^{n} \mathcal{W}_i\, Y_i\right)^2} \quad (18)$$

$$\hat{\theta}_{WOLS} = \exp[\hat{\alpha}_{\mathcal{W}}] \quad (19)$$

where,

$\hat{\alpha}_{\mathcal{W}} = \hat{\eta}_{WOLS} \sum_{i=1}^{n} \mathcal{W}_i\, Y_i - \sum_{i=1}^{n} \mathcal{W}_i\, \psi_i$, with $Y_i = \ln x_{(i)}$ and $\psi_i = \ln\left[-\ln\left(1 - \hat{F}_i\,\right)\right]$.

3. Loss function

The next subsections present three main types of loss function under study in this work.

3.1. Squared error loss function (SE)

Under the squared error loss function with the following form:

$$L(\hat{\vartheta}\,,\vartheta\,) = \left(\hat{\vartheta} - \vartheta\right)^2. \quad (20)$$

The Bayes estimator of ϑ, denoted by $\hat{\vartheta}_{SE}$ is given by,

$\hat{\vartheta}_{SE} = E(\vartheta|\underline{x})$.

3.2. General entropy loss function (GELF)

The General Entropy loss function for ϑ can be expressed as the following form (Calabria and Pulcini, 1996):

$$L(\hat{\vartheta},\vartheta) \propto (\hat{\vartheta}/\vartheta)^q - q\ln(\hat{\vartheta}/\vartheta) - 1 \quad ,q \neq 0 \tag{21}$$

where, $\hat{\vartheta}$ is an estimate of ϑ. The Bayes estimator of ϑ, denoted by $\hat{\vartheta}_{GE}$ is the value $\hat{\vartheta}$ which minimizes Eq. 21 and given as:

$$\hat{\vartheta}_{GE} = [E_\vartheta(\vartheta^{-q})]^{-\frac{1}{q}} \tag{22}$$

provided that $E_\vartheta(\vartheta^{-q})$ exists and finite.

3.3. Weighted general entropy loss function (WGELF)

The researcher proposes this loss function depending on weighted loss function General Entropy as following:

$$L_w(\hat{\vartheta},\vartheta) \propto w(\vartheta)\left[(\hat{\vartheta}/\vartheta)^q - q\ln(\hat{\vartheta}/\vartheta) - 1\right] \quad ,q \neq 0 \tag{23}$$

where L_w represents the estimated parameter that makes the expectation of loss function (Eq. 23) as smallest as possible. While, $w(\vartheta)$ represents the proposed weighted function, which equals to:

$$w(\vartheta) = \frac{1}{\vartheta^z}. \tag{24}$$

Depending on the posterior distribution of the parameter ϑ, and by using the proposed weighted function as in Eq. 24, we can get the estimated weighted Bayes of the parameter ϑ as the following:

$$E[L_w(\hat{\vartheta},\vartheta)] = \int_{\forall\vartheta} L_w(\hat{\vartheta},\vartheta) f(\vartheta|\underline{x})\,d\vartheta$$
$$= \int_{\forall\vartheta} w(\vartheta)\left[(\hat{\vartheta}/\vartheta)^q - q\ln(\hat{\vartheta}/\vartheta) - 1\right] f(\vartheta|\underline{x})\,d\vartheta$$
$$= \int_{\forall\vartheta} \frac{1}{\vartheta^z}\left[(\hat{\vartheta}/\vartheta)^q - q\ln(\hat{\vartheta}/\vartheta) - 1\right] f(\vartheta|\underline{x})\,d\vartheta$$
$$= \int_{\forall\vartheta} \frac{1}{\vartheta^z}(\hat{\vartheta}/\vartheta)^q f(\vartheta|\underline{x})\,d\vartheta - \int_{\forall\vartheta} \frac{q\ln\hat{\vartheta}}{\vartheta^z} f(\vartheta|\underline{x})\,d\vartheta + \int_{\forall\vartheta} \frac{q\ln\vartheta}{\vartheta^z} f(\vartheta|\underline{x})\,d\vartheta - \int_{\forall\vartheta} \frac{1}{\vartheta^z} f(\vartheta|\underline{x})\,d\vartheta$$
$$= \hat{\vartheta}^q \int_{\forall\vartheta} \frac{1}{\vartheta^{z+q}} f(\vartheta|\underline{x})\,d\vartheta - q\ln\hat{\vartheta}\int_{\forall\vartheta}\frac{1}{\vartheta^z} f(\vartheta|\underline{x})\,d\vartheta + q\int_{\forall\vartheta}\frac{\ln\vartheta}{\vartheta^z} f(\vartheta|\underline{x})\,d\vartheta - \int_{\forall\vartheta}\frac{1}{\vartheta^z} f(\vartheta|\underline{x})\,d\vartheta$$
$$E[L_w(\hat{\vartheta},\vartheta)] = \hat{\vartheta}^q E(\vartheta^{-(z+q)}|\underline{x}) - q\ln\hat{\vartheta}\,E(\vartheta^{-z}|\underline{x}) + qE\left(\frac{\ln\vartheta}{\vartheta^z}\Big|\underline{x}\right) - E(\vartheta^{-z}|\underline{x})$$
$$\frac{\partial L_w(\hat{\vartheta},\vartheta)}{\partial\hat{\vartheta}} = q\,\hat{\vartheta}^{q-1}E(\vartheta^{-(z+q)}|\underline{x}) - \frac{q}{\hat{\vartheta}}E(\vartheta^{-z}|\underline{x}) = 0.$$

So, we can find that:

$$q\,\hat{\vartheta}^{q-1}E(\vartheta^{-(z+q)}|\underline{x}) = \frac{q}{\hat{\vartheta}}E(\vartheta^{-z}|\underline{x}).$$

Consequently, the Bayesian estimation of the parameter ϑ using General. Entropy loss function will be:

$$\hat{\vartheta}_{WGE} = \frac{E(\vartheta^{-(z+q)}|\underline{x})}{E(\vartheta^{-z})|\underline{x})}. \tag{25}$$

Provided that $E_\vartheta(\vartheta^{-z})$ and $E_\vartheta(\vartheta^{-(z+q)})$ exists and finite, where E_φ denotes the expected value.

Note that, WGE loss function is a generalizing of the GE loss function, where GE is a special case of WGE when $z = 0$ in Eq. 25.

4. Bayesian methods of estimation of Weibull parameters

In this section, we derive the Bayes estimates of the scale and shape parameters of WD, and we assume the Jeffrey formula as a prior distribution for each parameter as follow (Sinha, 1986):

$$f(\eta) \propto \frac{1}{a} \quad ;0 \le a \le \infty \quad ,0 < \eta < a \tag{26}$$
$$f(\theta) \propto \frac{1}{\theta^c} \quad ;0 \le \theta \le \infty. \tag{27}$$

Therefore, the prior joint distribution function for parameters that we need to estimate are given as follow:

$$f(\eta,\theta) \propto \frac{1}{\theta^c} \quad ;\ c > 0\ ,0 \le \theta \le \infty. \tag{28}$$

By combining the likelihood function in Eq. 4 with the prior (p.d.f) of η and θ in Eq. 28, then the posterior distribution of η and θ is given by,

$$\pi(\eta,\theta) = \frac{f(\eta,\theta\,|\,\underline{x})\,f(\eta,\theta)}{\int_0^a\int_0^\infty f(\eta,\theta\,|\,\underline{x})\,f(\eta,\theta)\,d\theta\,d\eta}$$
$$= k\,\frac{\eta^n}{\theta^{n+c}}\left(\prod_{i=1}^n x_i\right)^{\eta-1}\exp\left[-\sum_{i=1}^n x_i^\eta/\theta\right]d\theta\,d\eta \tag{29}$$

where,

$$k^{-1} = \int_0^a\int_0^\infty \frac{\eta^n}{\theta^{n+c}}\left(\prod_{i=1}^n x_i\right)^{\eta-1}\exp\left[-\sum_{i=1}^n x_i^\eta/\theta\right]$$
$$= \Gamma(n+c-1)\int_0^a \frac{\eta^n\left(\prod_{i=1}^n x_i\right)^{\eta-1}}{\left(\sum_{i=1}^n x_i^\eta\right)^{n+c-1}}\,d\eta.$$

The marginal posterior *p.d.f* for parameter η can be obtained by integrating Eq. 19 with respect to θ as follow:

$$\pi_1^*(\eta\,|\,\underline{x}) = \int_{\forall\theta}\pi(\eta,\theta)\,d\theta$$
$$= \frac{\eta^n\prod_{i=1}^n x_i^{\eta-1}/\left(\sum_{i=1}^n x_i^\eta\right)^{n+c-1}}{\int_0^a \eta^n\prod_{i=1}^n x_i^{\eta-1}/\left(\sum_{i=1}^n x_i^\eta\right)^{n+c-1}d\eta} \quad ;0 < \eta < a. \tag{30}$$

The marginal posterior *p. d. f* for parameter θ can be obtained by integrating Eq. 29 with respect to η as follow:

$$\pi_2^*(\theta\,|\,\underline{x}) = \int_{\forall\eta}\pi(\eta,\theta)\,d\eta$$

$$= \frac{\frac{1}{\theta^{n+c}} \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1} \exp[-\sum_{i=1}^n x_i^{\eta}/\theta]\, d\eta}{\Gamma(n+c-1) \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta} \quad ; 0 < \theta < \infty. \quad (31)$$

4.1. Estimates based on squared error loss function (SELF)

By using the Squared Error Loss function in Eq. 20, the Bayes estimator $\hat{\eta}_{SE}$ for η, is given by,

$$\hat{\eta}_{SE} = E(\eta \mid \underline{x}) = \frac{\int_0^a \eta^{n+1} \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta}{\int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta} \quad (32)$$

$$\hat{\theta}_{SE} = E(\theta \mid \underline{x}) = \frac{\int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-2} d\eta}{(n+c-2) \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta}. \quad (33)$$

4.2. Estimates based on general entropy loss function (GELF)

By using the General Entropy Loss function in Eq. 21, the Bayes estimator $\hat{\eta}_{GE}$ for η, is given by,

$$\hat{\eta}_{GE} = E[(\eta^{-q} \mid \underline{x})]^{-\frac{1}{q}} = [\int_0^a \eta^{-q} \pi_1^*(\eta \mid \underline{x})\, d\eta]^{-\frac{1}{q}} = \left[\frac{\int_0^a \eta^{n-q} \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta}{\int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta}\right]^{-\frac{1}{q}} \quad (34)$$

and the Bayes estimator $\hat{\theta}_{GE}$ for θ, is given by,

$$\hat{\theta}_{GE} = E[(\theta^{-q} \mid \underline{x})]^{-\frac{1}{q}} = [\int_0^a \theta^{-q} \pi_2^*(\theta \mid \underline{x})\, d\theta]^{-\frac{1}{q}} = \left[\frac{\Gamma(q+n+c-1)}{\Gamma(n+c-1)} \frac{\int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{q+n+c-1} d\eta}{\int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta}\right]^{-\frac{1}{q}} \quad (35)$$

4.3. Estimates based on weighted general entropy loss function (WGELF)

Under the weighted General Entropy loss function, and by using 23, the Bayes estimator $\hat{\eta}_{WGE}$ for η is given by,

$$\hat{\eta}_{WGE} = \left[\frac{E(\eta^{-(z+q)} \mid \underline{x})}{E(\eta^{-z}) \mid \underline{x})}\right]^{-\frac{1}{q}} = \left[\frac{I_1}{I_2}\right]^{-\frac{1}{q}} \quad (36)$$

where,

$$I_1 = \frac{1}{I_3} \int_0^a \eta^{n-(z+q)} \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta \quad (37)$$

and,

$$I_2 = \frac{1}{I_3} \int_0^a \eta^{n-z} \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta \quad (38)$$

$$I_3 = \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta \quad (39)$$

Thus the weighted Bayes estimator for the shape parameter η is:

$$\hat{\eta}_{WGE} = \left[\frac{I_1}{I_2}\right]^{-\frac{1}{q}} = \left[\frac{\int_0^a \eta^{n-(z+q)} \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta}{\int_0^a \eta^{n-z} \prod_{i=1}^n x_i^{\eta-1}/(\sum_{i=1}^n x_i^{\eta})^{n+c-1} d\eta}\right]^{-\frac{1}{q}} \quad (40)$$

and the Bayes estimator $\hat{\theta}_{WGE}$ for θ is given by,

$$\hat{\theta}_{WGE} = \left[\frac{E(\eta\theta^{-(z+q)} \mid \underline{x})}{E(\theta^{-z}) \mid \underline{x})}\right]^{-\frac{1}{q}} = \left[\frac{I_4}{I_5}\right]^{-\frac{1}{q}} \quad (41)$$

where,

$$I_4 = \frac{1}{\Gamma(n+c-1)} I_3 \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1} \left[\int_0^\infty \frac{1}{\theta^{n+c+z+q}} \exp(-\sum_{i=1}^n x_i^{\eta}/\theta)\, d\theta\right] d\eta = \frac{\Gamma(n+c+z+q-1) \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(x_i^{\eta})^{n+c+z+q-1}}{\Gamma(n+c-1).I_3} \quad (42)$$

and,

$$I_5 = \frac{1}{\Gamma(n+c-1)} I_3 \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1} \left[\int_0^\infty \frac{1}{\theta^{n+c+z}} \exp(-\sum_{i=1}^n x_i^{\eta}/\theta)\, d\theta\right] d\eta = \frac{\Gamma(n+c+z-1) \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(x_i^{\eta})^{n+c+z-1}}{\Gamma(n+c-1).I_3} \quad (43)$$

Thus the weighted Bayes estimator for the shape parameter θ is:

$$\hat{\theta}_{WGE} = \left[\frac{I_4}{I_5}\right]^{-\frac{1}{q}} = \left[\frac{\Gamma(n+c+z+q-1) \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(x_i^{\eta})^{n+c+z+q-1}}{\Gamma(n+c+z-1) \int_0^a \eta^n \prod_{i=1}^n x_i^{\eta-1}/(x_i^{\eta})^{n+c+z-1}}\right]^{-\frac{1}{q}} \quad (44)$$

5. Simulation

In this section, Monte–Carlo simulation is employed to compare the performance of four estimates (MLE, OLS, WOLS, and Bayes Estimators under different loss function including SE, GE, and WGE) for unknown shape and scale parameters based on the mean squared errors (MSE's) as follows:

Step (1): Setting the default values:

a. To observe the effect the parameters of Weibull distribution on the estimates, five different values of (η, θ) were selected as case1 (0.5, 0.5), case2 (0.5, 1.5), case3 (1.5, 0.5), case4 (1.5, 1.5) and case 5 (1.5, 2.5).
b. Choose the sample size $n = 15, 25, 50, 75$ and 100.
c. The values of GELF's constant (q) selected to be (q =1.5 and -1.5) the positive and negative values were selected to represent both cases of upper estimate and under estimate, respectively.
d. The values of WGELF's constant (z) were selected to be (z =-2 and 2).
e. The values of the constant c in the Jeffery formula was selected$c = 3$.
f. The number of iteration (L) was chosen to be (10000).

Step (2): Generate the sample random values of WD by using the formula:

$$x_i = [-\theta\, Log\, (u(1))]^{\frac{1}{\eta}}$$

where u=$RND(1)$ is a random variable distributed as a uniform distribution for the period (0,1).

Step (3): Calculating the ML, OLS. WOLS and Bayesian estimator of the parameters function of (WD) according to the formulas that had been obtained we calculate $\vartheta = (\hat{\eta}\,,\,\hat{\theta})$.

Step (4): Comparing the different estimation methods according to the values of mean squared error (MSE), where:

$$MSE = \frac{\sum_{i=1}^{L}(\hat{\vartheta}_i - \vartheta)^2}{L}$$

where $\hat{\vartheta}_i$ is the estimate at the i^{th} run.

The simulation results of MSE are tabulated in Tables 1-2.

Table 1: MSEs of the estimates of η with different cases

(θ,η)	n	$\hat{\eta}_{ML}$	$\hat{\eta}_{OLS}$	$\hat{\eta}_{WLS}$	$\hat{\eta}_{SE}$	$\hat{\eta}_{GE}$		$\hat{\eta}_{WGE}$			
								$Z=2$		$Z=-2$	
						q		q			
						-1.5	1.5	-1.5	1.5	-1.5	1.5
	15	0.0201	0.0164	0.0145	0.0245	0.0258	0.0216	0.0172	0.0154	0.0436	0.0316
	25	0.0093	0.0099	0.0088	0.0103	0.0107	0.0094	0.0081	0.0076	0.0151	0.0127
(0.5,0.5)	50	0.0035	0.0051	0.0042	0.0038	0.0039	0.0035	0.0033	0.0032	0.0051	0.0043
	75	0.0022	0.0036	0.0029	0.0024	0.0024	0.0022	0.0022	0.0021	0.0028	0.0026
	100	0.0016	0.0027	0.0022	0.0017	0.0017	0.0016	0.0016	0.0016	0.0019	0.0018
	15	0.0288	0.0165	0.0143	0.0169	0.0177	0.0119	0.0124	0.0113	0.0226	0.0162
	25	0.0089	0.0098	0.0089	0.0084	0.0087	0.0063	0.0069	0.0063	0.0098	0.0080
(1.5,0.5)	50	0.0034	0.0052	0.0042	0.0033	0.0034	0.0030	0.0030	0.0030	0.0038	0.0033
	75	0.0022	0.0035	0.0029	0.0022	0.0022	0.0020	0.0020	0.0020	0.0023	0.0022
	100	0.0016	0.0027	0.0021	0.0016	0.0016	0.0015	0.0015	0.0015	0.0017	0.0016
	15	0.1901	0.1489	0.1366	0.2116	0.2223	0.1822	0.1505	0.1322	0.3644	0.2812
	25	0.0850	0.0920	0.0833	0.0962	0.0998	0.0852	0.0762	0.0688	0.1390	0.1120
(0.5,1.5)	50	0.0316	0.0467	0.0378	0.0347	0.0354	0.0321	0.0304	0.0288	0.0449	0.0377
	75	0.0199	0.0313	0.0251	0.0213	0.0216	0.0205	0.0193	0.0189	0.0253	0.0225
	100	0.0142	0.0241	0.0192	0.0149	0.0151	0.0145	0.0139	0.0137	0.0177	0.0163
	15	0.1782	0.1491	0.1332	0.1518	0.1593	0.1030	0.1135	0.0988	0.1900	0.1420
	25	0.0787	0.0883	0.0774	0.0640	0.0660	0.0581	0.0554	0.0569	0.0866	0.0704
(1.5,1.5)	50	0.0323	0.0463	0.0389	0.0293	0.0297	0.0272	0.0274	0.0272	0.0334	0.0293
	75	0.0201	0.0320	0.0259	0.0188	0.0190	0.0180	0.0180	0.0179	0.0209	0.0191
	100	0.0146	0.0241	0.0195	0.0139	0.0140	0.0136	0.0134	0.0136	0.0150	0.0142
	15	0.9007	0.1529	0.1374	0.0984	0.1013	0.0883	0.0911	0.1028	0.1354	0.1103
	25	0.1335	0.0908	0.0798	0.0559	0.0570	0.0515	0.0534	0.0568	0.0699	0.0612
(1.5,2.5)	50	0.0305	0.0447	0.0369	0.0259	0.0261	0.0261	0.0256	0.0274	0.0301	0.0272
	75	0.0199	0.0313	0.0259	0.0177	0.0178	0.0176	0.0174	0.0182	0.0195	0.0272
	100	0.0144	0.0242	0.0195	0.0132	0.0132	0.0130	0.0131	0.0133	0.0145	0.0185

Table 2: MSEs of the estimates of θ with different cases

$(\theta\,,\eta)$	n	$\hat{\theta}_{ML}$	$\hat{\theta}_{OLS}$	$\hat{\theta}_{WLS}$	$\hat{\theta}_{SE}$	$\hat{\theta}_{GE}$		$\hat{\theta}_{WGE}$			
								$Z=2$		$Z=-2$	
						q		q			
						-1.5	1.5	-1.5	1.5	-1.5	1.5
	15	0.0226	0.0268	0.0242	0.0218	0.0221	0.0250	0.0236	0.0326	0.0261	0.0218
	25	0.0133	0.0161	0.0144	0.0119	0.0120	0.0137	0.0127	0.0172	0.0133	0.0121
(0.5,0.5)	50	0.0054	0.0084	0.0071	0.0055	0.0056	0.0059	0.0058	0.0069	0.0060	0.0054
	75	0.0036	0.0058	0.0048	0.0037	0.0037	0.0038	0.0038	0.0043	0.0037	0.0036
	100	0.0027	0.0043	0.0036	0.0027	0.0027	0.0028	0.0027	0.0031	0.0028	0.0027
	15	0.4008	0.3163	0.2931	0.4733	0.5515	0.2062	0.2217	0.2301	1.1451	0.4352
	25	0.1903	0.1543	0.1583	0.2006	0.2184	0.1207	0.1319	0.1332	0.3153	0.1779
(1.5,0.5)	50	0.0741	0.0654	0.0688	0.0757	0.0788	0.0617	0.0629	0.0643	0.0965	0.0722
	75	0.0479	0.0438	0.0466	0.0487	0.0500	0.0420	0.0428	0.0431	0.0553	0.0471
	100	0.0341	0.0317	0.0345	0.0345	0.0352	0.0313	0.0315	0.0321	0.0408	0.0337
	15	0.0202	0.0274	0.0253	0.0226	0.0229	0.0247	0.0240	0.0323	0.0256	0.0219
	25	0.0109	0.0164	0.0145	0.0119	0.0120	0.0131	0.0126	0.0165	0.0129	0.0122
(0.5,1.5)	50	0.0054	0.0085	0.0070	0.0055	0.0056	0.0059	0.0058	0.0069	0.0059	0.0055
	75	0.0035	0.0056	0.0046	0.0036	0.0036	0.0038	0.0037	0.0043	0.0036	0.0036
	100	0.0027	0.0044	0.0035	0.0027	0.0027	0.0029	0.0028	0.0032	0.0028	0.0027
	15	0.3796	0.2885	0.2782	0.4247	0.4891	0.2027	0.2124	0.2300	0.9180	0.4041
	25	0.1932	0.1531	0.1546	01519	0.1617	0.1213	0.1225	0.1331	0.3033	0.1828
(1.5,1.5)	50	0.0759	0.0664	0.0710	0.0678	0.0698	0.0634	0.0615	0.0668	0.0952	0.0745
	75	0.0475	0.0431	0.0464	0.0442	0.0450	0.0416	0.0415	0.0431	0.0583	0.0463
	100	0.0339	0.0316	0.0337	0.0321	0.0326	0.0313	0.0308	0.0318	0.0399	0.0354
	15	1.4701	1.7555	1.5130	1.1615	1.3806	0.7074	0.6919	0.8723	7.2648	2.0835
	25	0.8370	0.7375	0.7094	0.6127	0.6764	0.4367	0.4451	0.5088	1.9262	0.8874
(1.5,2.5)	50	0.3335	0.2900	0.2997	0.2726	0.2847	0.2455	0.2380	0.2619	0.4910	0.3347
	75	0.2077	0.1907	0.2004	0.1807	0.1863	0.1680	0.1624	0.1741	0.2630	0.1998
	100	0.1482	0.1430	0.1475	0.1340	0.1370	0.1250	0.1243	0.1282	0.1845	0.1493

6. Concluding remarks

From the results in the above Tables 1 and 2, we can state the following points:

1. Table 1 shows that the performance of the Bayesian estimator of the shape parameter under WGE loss is the best estimator comparing to the other estimators for all cases of (η, θ) and all sample sizes n. On the other hand, the performance of the WLS estimator of the shape parameter is the best estimator comparing to the MLE or OLS. The results also show that MSE's of all estimators of shape parameter is decreasing with the increase of the value of the scale parameter.
2. Table 2 shows that the performance of the Bayesian estimator of the scale parameter under WGE loss function with $(\eta = 0.5, \theta = 0.5)$ $(\eta = 1.5, \theta = 0.5)$ and $(\eta = 1.5, \theta = 2.5)$ is the best estimator comparing to the other estimators for most sample sizes. While the performance of the Bayesian estimator of the scale parameter under GE loss function with $(\eta = 0.5, \theta = 1.5)$ and $(\eta = 1.5, \theta = 1.5)$ is the best estimator comparing to the other estimators for all sample sizes. On the other hand, the performance of the MLE estimator of the scale parameter with $(\eta = 0.5, \theta = 0.5)$ and $(\eta = 1.5, \theta = 0.5)$ is the best estimator comparing to the OLS or WLS, While the performance of the WLS estimator, with $(\eta = 0.5, \theta = 1.5)$ $(\eta = 1.5, \theta = 1.5)$ and $(\eta = 1.5, \theta = 2.5)$ is the best estimator comparing to the MLE or OLS. The results also show that MSE's of all estimates of scale parameter is increasing for an increase of the Parameter value with all sample sizes.
3. The results showed that the values of all MSE's decrease as n increases.

7. Conclusion

In this work, we developed the GE loss function to estimate the parameters of WD. The development was through merging a weight into GE to produce a new loss function called WGE. Then WGE was used to derive parameters of the WD.

Furthermore, we conducted a Monte Carlo simulation to examine the performance of the proposed method WGE. Then we compared the proposed method with other methods, including SE, GE, MLE, OLS, and WLS. The results show that the performance of the Bayes estimator under developed method (WGE) loss function is the best for estimating shape parameters in all cases and has good performance for estimating scale parameter.

Acknowledgment

This Publication was supported by the Deanship of Scientific Research at Prince Sattam bin Abdulaziz University.

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

References

Al-Duais F (2020). Using non-linear programming to determine weighted coefficient of balanced loss function for estimating parameters and reliability function of weibull distribution. Pushpa Publishing House, Allahabad, 62(1): 31-53. https://doi.org/10.17654/AS062010031

Al-Duais F and Alhagyan M (2020). Bayesian estimates based on record values under weighted LINEX loss function. Pakistan Journal of Statistics and Operation Research, 16(1): 11-19. https://doi.org/10.18187/pjsor.v16i1.2854

Aslam M, Kazmi SMA, Ahmad I, and Shah SH (2014). Bayesian estimation for parameters of the Weibull distribution. Science International (Lahore), 26(5): 1915-1920.

Basumatary H, Sreevalsan E, and Sasi KK (2005). Weibull parameter estimation-A comparison of different methods. Wind Engineering, 29(3): 309-315. https://doi.org/10.1260/030952405774354895

Calabria R and Pulcini G (1996). Point estimation under asymmetric loss functions for left-truncated exponential samples. Communications in Statistics-Theory and Methods, 25(3): 585-600. https://doi.org/10.1080/03610929608831715

Gupta PK and Singh AK (2017). Classical and Bayesian estimation of Weibull distribution in presence of outliers. Cogent Mathematics, 4(1): 1300975. https://doi.org/10.1080/23311835.2017.1300975

Guure CB, Ibrahim NA, Adam MB, Bosomprah S, and Ahmed AO (2014). Bayesian parameter and reliability estimate of Weibull failure time distribution. The Bulletin of the Malaysian Mathematical Sciences Society, Series 2, 37(2): 611–632.

Hung WL and Liu YC (2004). Estimation of Weibull parameters using a fuzzy least-squares method. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 12(05): 701-711. https://doi.org/10.1142/S0218488504003156

Marks NB (2005). Estimation of Weibull parameters from common percentiles. Journal of Applied Statistics, 32(1): 17-24. https://doi.org/10.1080/0266476042000305122

Mohammed AAO and Ibrahim NA (2011). Bayesian survival estimator for Weibull distribution with censored data. Journal of Applied Sciences, 11(2): 393-396. https://doi.org/10.3923/jas.2011.393.396

Nwobi FN and Ugomma CA (2014). A comparison of methods for the estimation of Weibull distribution parameters. Metodoloski Zvezki, 11(1): 65-78.

Pobočíková I and Sedliačková Z (2014). Comparison of four methods for estimating the Weibull distribution parameters. Applied Mathematical Sciences, 8: 4137-4149. https://doi.org/10.12988/ams.2014.45389

Saeed MK, Salam A, Rehman AU, and Saeed MA (2019). Comparison of six different methods of Weibull distribution for wind power assessment: A case study for a site in the Northern region of Pakistan. Sustainable Energy Technologies and Assessments, 36: 100541. https://doi.org/10.1016/j.seta.2019.100541

Sinha SK (1986). Reliability and life testing. John Wiley and Sons, Hoboken, USA.