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On the H-space and the product of two H-spaces

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ABSTRACT

Considering the set of real numbers R, for each x ∈ A, B(x)={(x−ϵ,x+ϵ):ϵ>0}, and for each x∈ R\A, let B(x)={[x,x+ϵ):ϵ>0}, the unique topology generated by {B(x): x ∈ R} is denoted by τ(A) and (R, τ(A)) is called an H-space. In this paper, we give some results about these spaces and the product of two of them, including the separation axioms, wD property, various types of compactness and connectedness, and weaker properties of normality.

1. Introduction

Hattori (2010) defined on the set of real numbers $\mathbb{R}$ topologies that lie between the usual topology and the Sorgenfrey topology; $\mathbb{R}$ with these topologies were called H-spaces. These spaces were previously studied in Bouziad and Sukhacheva (2017), Chatyrko and Hattori (2016; 2013), and Kulesza (2017). In this paper, we give some results about these spaces and the product of two of them. Throughout this paper, we denote the set of positive integers by $\mathbb{N}$, the rationals by $\mathbb{Q}$, the irrationals by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space and a Tychonoff ($T_{3\frac{1}{2}}$) space is a T_1 completely regular space. We do not assume T_2 in the definition of compactness, paracompactness and countable compactness. We do not assume regularity in the definition of Lindelöfness. For a subset A of a space X, $intA$ and $\overline{A}$ denote the interior and the closure of A, respectively.

Recall that the *Sorgenfrey topology* $\mathcal{S}$ on $\mathbb{R}$ is the unique topology generated by the base $\{[x,y): x < y; x,y \in \mathbb{R}\}$. $\mathbb{R}$ with this topology $\mathcal{S}$ is called the *Sorgenfrey line.* Let $\mathcal{U}$ denote the usual metric topology on $\mathbb{R}$.

Definition 1.1: *Let A be any subset of $\mathbb{R}$. For each $x \in A$, let $\mathcal{B}(x) = \{(x-\epsilon, x+\epsilon): \epsilon > 0\}$. For each $x \in \mathbb{R}\backslash A$, let $\mathcal{B}(x) = \{[x, x+\epsilon): \epsilon > 0\}$. The unique topology generated by $\{\mathcal{B}(x): x \in \mathbb{R}\}$ is denoted by $\tau(A)$ and $(\mathbb{R}, \tau(A))$ is called an H-space* (Bouziad and Sukhacheva, 2017; Chatyrko and Hattori, 2016; 2013; Kulesza, 2017).

Observe that if we interchange the local bases in Definition 1.1 and define a new topology $\mathcal{S}_A$ on $\mathbb{R}$, then for any subset A of $\mathbb{R}$, we have $\mathcal{S}_A = \tau(\mathbb{R}\backslash A)$. Note that if $A = \emptyset$, then $\tau(A) = \mathcal{S}$ and if $A = \mathbb{R}$, then $\tau(A) = \mathcal{U}$. From now on, when we consider an H-topology $\tau(A)$ on $\mathbb{R}$, we are assuming that A is a non-empty proper subset of $\mathbb{R}$. Observe that if $x \in \mathbb{R}\backslash A$ and $[x,y) \in \mathcal{B}(x)$, then $[x,y)$ need not be clopen (closed-and-open) because y could be an element of A. But if $x, y \in \mathbb{R}\backslash A$, then $[x,y)$ is clopen. It is clear that for any subset A of $\mathbb{R}$ we have that $\mathcal{U}$ is coarser than $\tau(A)$ and $\tau(A)$ is coarser than $\mathcal{S}$, i.e., $\mathcal{U} \subset \tau(A) \subset \mathcal{S}$, (Hattori, 2010). So, it is clear that every H-space is completely Hausdorff. Any H-space is first countable, hence Fréchet, sequential, and of countable tightness, (Engelking, 1977).

Proposition 1.2: *For any subset A of $\mathbb{R}$, $(\mathbb{R}, \tau(A))$ is regular.*

Observe that the Sorgenfrey line is Lindelöf and $\tau(A) \subset \mathcal{S}$, so any H-space $(\mathbb{R}, \tau(A))$ is Lindelöf. Since any Lindelöf regular space is paracompact and normal, then every H-space is paracompact and normal. Furthermore, the H-space $(\mathbb{R}, \tau(A))$ is perfectly normal, hence T_6.

Theorem 1.3: *For any subset A of $\mathbb{R}$, the H-space $(\mathbb{R}, \tau(A))$ is perfectly normal, hence T_6.*

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Proof: We show that every open set is an F_σ-set. Let $U \in \tau(A)$ be arbitrary. Without loss of generality, assume that $U \neq \emptyset$. Let $V = \text{int}_{\mathcal{U}} U$, the interior of U with respect to the usual topology. Then, V is an F_σ-set in $\mathcal{U}$, because $(\mathbb{R}, \mathcal{U})$ is perfectly normal. Hence, $V = \bigcup_{n\in\mathbb{N}} F_n$, where F_n is closed in $(\mathbb{R}, \mathcal{U})$ for each $n \in \mathbb{N}$. Since $\mathcal{U} \subset \tau(A)$, then F_n is also closed in $(\mathbb{R}, \tau(A))$. Then, V an F_σ-set in $\tau(A)$.

Claim 1: The set $B = U\backslash V$ is a subset of $\mathbb{R}\backslash A$.

Proof of Claim 1: If $B = \emptyset$, we are done. Assume that $B \neq \emptyset$. Pick $x \in B$ arbitrary. Then $x \notin V = \text{int}_{\mathcal{U}} U$ and $x \in U$. Suppose $x \in A$, and then there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$ because U is open in $\tau(A)$. Thus $x \in \text{int}_{\mathcal{U}} U$ which means that $x \in V$ and this is a contradiction. Therefore, $B = U\backslash V \subseteq \mathbb{R}\backslash A$ and Claim 1 is proved. Hence, for each $a \in B$, there exists $y_a \in \mathbb{R}$ such that $[a, y_a) \subseteq U$.

Claim 2: For distinct $a, b \in B$, $[a, y_a) \cap [b, y_b) = \emptyset$.

Proof of claim 2: Assume, on the contrary, that there exist distinct $a, b \in B$ with $[a, y_a) \cap [b, y_b) \neq \emptyset$. Then, $a \in [b, y_b)$ or $b \in [a, y_a)$. Since $a \neq b$, this implies $a \in (b, y_b) \subseteq U$ or $b \in (a, y_a) \subseteq U$. Hence, $a \in V$ or $b \in V$, a contradiction. Thus, claim 2 is proved.

Now, $\{(a, y_a): a \in B\}$ is a pairwise disjoint family of non-empty open intervals. Since any H-space is separable, we conclude that B is countable. For each $a \in B$, put $x_a = \frac{a+y_a}{2}$. Then, $[a, x_a] \subset [a, y_a) \subseteq U$. We actually have $U = \bigcup_{a\in B} [a, x_a] \cup V$ is a union of two F_σ-sets, and hence is an F_σ-set.

It follows that the H-space $(\mathbb{R}, \tau(A))$ is hereditarily normal, hereditarily paracompact, and hereditarily Lindelöf, as stated in Chatyrko and Hattori (2013).

Recall that a space X is called *Dowker* if X is T_4 and $X \times I$ is not normal where $I = [0,1]$ is considered with its usual metric topology. Since the product of a Lindelöf space with a compact space is Lindelöf (Engelking, 1977), then for any subset A of $\mathbb{R}$, $(\mathbb{R}, \tau(A)) \times (I, \mathcal{U}_I)$ is Lindelöf. Since the product is also T_3, then it is normal. Thus we conclude the following theorem.

Theorem 1.4: *Every H-space $(\mathbb{R}, \tau(A))$ is not Dowker.*

Recall that a space X is said to satisfy *property wD* (Nyikos, 1981) if for every infinite closed discrete subspace C of X, there exists a discrete family $\{U_n: n \in \mathbb{N}\}$ of open subsets of X such that each U_n meets C at exactly one point. That is, for each $n \in \mathbb{N}$ we have $U_n \cap C$ is a singleton.

Theorem 1.5: *Every H-space $(\mathbb{R}, \tau(A))$ is wD.*

Proof: Let C be any infinite closed discrete subspace of $(\mathbb{R}, \tau(A))$. Since $(\mathbb{R}, \tau(A))$ is separable and normal, then, by Jones' Lemma, we conclude that C is countably infinite. Pick an element $c_1 \in C$, then c_1 has only two cases, the set $\{c \in C: c < c_1\}$ is infinite or $\{c \in C: c > c_1\}$ is infinite. Without loss of generality, assume that the set $\{c \in C: c > c_1\}$ is infinite. Denote the set $\{c \in C: c > c_1\} \cup \{c_1\}$ by C'. Observe that $C' \subseteq C$. Since $\mathbb{R}$ is ordered by the relation $<$ and C' is countably infinite, we can rearrange and enumerate the elements of C' to make it an increasing sequence $C' = (c_n)_{n\in\mathbb{N}}$ such that $c_n < c_{n+1}$ for each $n \in \mathbb{N}$. If $c_n \longrightarrow x$ in the space $(\mathbb{R}, \tau(A))$, then $c_n \longrightarrow x$ in the space $(\mathbb{R}, \mathcal{U})$, i.e., C' is bounded above and $x = \sup C'$. Since C is closed, then $x \in C$, and there exists an open neighborhood U of x such that $C \cap U = \{x\}$, because C is discrete. Hence, $C' \cap U = \{x\}$. This contradicts that $c_n \longrightarrow x$, as c_n is increasing. Thus, the sequence $C' = (c_n)_{n\in\mathbb{N}}$ is divergent in $(\mathbb{R}, \tau(A))$. Now, for each $n \in \mathbb{N}$, let,

$$\epsilon_n = \frac{c_{n+1} - c_n}{4}.$$

For each $n \in \mathbb{N}$, put

$$U_n = \begin{cases} (c_n - \epsilon_n, c_n + \epsilon_n) & ; \ if \quad c_n \in A \\ [c_n, c_n + \epsilon_n) & ; \ if \quad c_n \in \mathbb{R}\backslash A \end{cases}$$

By our construction, it is clear that the family $\{U_n: n \in \mathbb{N}\}$ is a discrete family in $(\mathbb{R}, \tau(A))$ consisting of open sets such that $U_n \cap C = \{c_n\}$ for each $n \in \mathbb{N}$. Therefore, $(\mathbb{R}, \tau(A))$ satisfies property wD.

Observe that $\{(-n, n): a \in \mathbb{N}\}$ is an open cover for $(\mathbb{R}, \tau(A))$ that has no finite subcover, so $(\mathbb{R}, \tau(A))$ is not compact nor countably compact. Recall that a space X is *pseudo compact* if X is a Tychonoff and every continuous real-valued function defined on X is bounded. In a T_4 space, this is equivalent to countable compactness (Engelking, 1977). Hence, for any choice of $A \subset \mathbb{R}$ the H-space $(\mathbb{R}, \tau(A))$ is neither compact, countably compact, nor pseudo compact.

Let us consider the compact subsets in any H-space, $\tau(A)$. By the relation $\mathcal{U} \subset \tau(A) \subset \mathcal{S}$, we know that any compact subset C of $\tau(A)$ must be compact in $\mathcal{U}$, so it is closed and bounded (in the sense of the usual metric). Moreover, any subset C that is compact in $\mathcal{S}$ must be compact in $\tau(A)$, and these are exactly the bounded $\mathcal{S}$-closed subsets that contain no strictly increasing sequence (Espelie and Joseph, 1976). Hence, it is natural to ask for a characterization of compact subsets of $\tau(A)$. Such characterization must depend on A, of course.

It is clear that a subset of A is compact in $\tau(A)$ if and only if it is compact in $\mathcal{U}$, and a subset of $\mathbb{R}\backslash A$ is compact in $\tau(A)$ if and only if it is compact in $\mathcal{S}$. By taking finite unions of compact subsets of A and $\mathbb{R}\backslash A$, one can trivially construct some compact subsets of $\tau(A)$. While we are far from having a complete characterization of compact subsets of an H-space, we give an example of a compact subset that cannot be constructed in this way.

Example 1.6: *Let* $A = (-\infty, 1]$, *and consider the set* $C = [-3,-2] \cup \{1\} \cup \{1 + \frac{1}{n} : n \in \mathbb{N}\}$ *in the space* $(\mathbb{R}, \tau(A))$. *This is a compact subset. Note that* $C \cap A$ *is compact in* $\mathcal{U}$, *yet* $C \cap \mathbb{R}\backslash A$ *is not compact in* $\mathcal{S}$, *as it is not closed.*

A space X is *totally imperfect* if every compact subspace of X is countable. For example, the Sorgenfrey line is totally imperfect, while the real line is not. The following was proved in Bouziad and Sukhacheva (2017): "Space $(\mathbb{R}, \tau(A))$ is totally imperfect if and only if A is totally imperfect". This leads to the following results.

Corollary 1.7: *If* A *is totally imperfect, then* $(\mathbb{R}, \tau(A))$ *is not* σ*-compact.*

Recall that a space X is *sequentially compact* if X is Hausdorff and any sequence of elements of X has a convergent subsequence. Observe that since $\mathcal{U} \subset \tau(A)$ for any $\emptyset \neq A \subset \mathbb{R}$ and both are Hausdorff, then if a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in $(\mathbb{R}, \tau(A))$, then $(x_n)_{n\in\mathbb{N}}$ also converges to x in $(\mathbb{R}, \mathcal{U})$. Thus if $x_n \nrightarrow x$ in $(\mathbb{R}, \mathcal{U})$, then $x_n \nrightarrow x$ also in $(\mathbb{R}, \tau(A))$. Any H-space $(\mathbb{R}, \tau(A))$ is not sequentially compact, and here is a counterexample.

Example 1.8: *Let* A *be any non-empty proper subset of* $\mathbb{R}$. *Pick* $a \in \mathbb{R}\backslash A$. *For each* $n \in \mathbb{N}$, *pick* $a_n \in (a - \frac{1}{n}, a - \frac{1}{n+1})$. *Then the sequence* $(a_n)_{n\in\mathbb{N}}$ *is an increasing sequence converges in* $(\mathbb{R}, \mathcal{U})$ *to* a. *Then, any subsequence of* $(a_n)_{n\in\mathbb{N}}$ *must converge to* a *in* $(\mathbb{R}, \mathcal{U})$, *and if a subsequence of* $(a_n)_{n\in\mathbb{N}}$ *converges in* $(\mathbb{R}, \tau(A))$, *then it must also converge to* a. *Since* $a \in \mathbb{R}\backslash A$, *then any basic open neighborhood of* a *is of the form* $[a, a + \epsilon)$, *where* $\epsilon > 0$. *Since* $[a, a + \epsilon)$ *contains no elements of the sequence* $(a_n)_{n\in\mathbb{N}}$, *we conclude no subsequence of* $(a_n)_{n\in\mathbb{N}}$ *is convergent to* a. *Thus in* $(\mathbb{R}, \tau(A))$, *the sequence* $(a_n)_{n\in\mathbb{N}}$ *has no convergent subsequence. Therefore,* $(\mathbb{R}, \tau(A))$ *is not sequentially compact.* .1*in*. 1*in.*

2. Product of two *H*-spaces

Let A and B be any subsets of $\mathbb{R}$. Observe that we always have that $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is separable, just take $D = \mathbb{Q} \times \mathbb{Q}$, which is a countable dense subspace. Since $(\mathbb{R}, \tau(A))$ is first countable for any A, then $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is first countable. Kulesza (2017) stated that "Space $(\mathbb{R}, \tau(A))$ is second countable if and only if it is metrizable if and only if the set $\mathbb{R}\backslash A$ is countable". Hence, we may deduce the following.

Theorem 2.1: For any subsets A and B of $\mathbb{R}$, the following statements are equivalent:

1. The sets $\mathbb{R}\backslash A$ and $\mathbb{R}\backslash B$ are both countable.
2. Space $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is second countable.
3. Space $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is metrizable.

Proof: If $\mathbb{R}\backslash A$ and $\mathbb{R}\backslash B$ are countable, then $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is a product of two-second countable spaces and must be second countable. If $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is second countable, then $(\mathbb{R}, \tau(A))$ and $(\mathbb{R}, \tau(B))$ are secondly countable, hence metrizable. Then, $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is metrizable. If $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is metrizable, then both $(\mathbb{R}, \tau(A))$ and $(\mathbb{R}, \tau(B))$ are metrizable. Hence, $\mathbb{R}\backslash A$ and $\mathbb{R}\backslash B$ are countable.

Theorem 2.2: *For any subsets* A *and* B *of* $\mathbb{R}$, *space* $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ *is neither compact nor countably compact, nor pseudocompact.*

Following Corollary 1.7, we have the next result.

Corollary 2.3: *If* A *is totally imperfect, then* $(\mathbb{R}^2, \tau(A))$ *is not* σ*-compact.*

When it comes to local compactness, it was also proved in Bouziad and Sukhacheva (2017) that " The H-space $(\mathbb{R}, \tau(A))$ is locally compact if and only if $\mathbb{R}\backslash A$ is closed in $(\mathbb{R}, U)$ and discrete in $(\mathbb{R}, \mathcal{S})$". This can be generalized to a finite product. However, it is important to note that this is not the case for an infinite product of H-spaces. By Engelking (1977), Theorem 3.3.13, we can deduce the following.

Theorem 2.4: *Space* $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ *is locally compact if and only if* $A, B \subset \mathbb{R}$ *such that* $\mathbb{R}\backslash A$ *and* $\mathbb{R}\backslash B$ *are closed in* $(\mathbb{R}, U)$ *and discrete in* $(\mathbb{R}, \mathcal{S})$.

By Engelking (1977), Theorem 2.3.11, $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is T_i where $i \in \{0,1,2,2\frac{1}{2}, 3, 3\frac{1}{2}\}$. There are some cases where $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ cannot be normal. In Chatyrko and Hattori (2013) Theorem 2.1, it was proved that " *if* A *is a closed countable subset of* $\mathbb{R}$, *then* $(\mathbb{R}, \tau(A)) \cong (\mathbb{R}, \mathcal{S})$"; this result was further strengthened by Kulesza (2017) (Theorem 6) to give the following characterization: "$(\mathbb{R}, \tau(A))$ is homeomorphic to the Sorgenfrey line if and only if A is scattered in $(\mathbb{R}, \mathcal{U})$". Since $(\mathbb{R}^2, S)$ is not normal (Steen et al., 1978), we reach the following result.

Theorem 2.5: *If* A *and* B *are scattered in* $(\mathbb{R}, \mathcal{U})$, *then* $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ *is not normal.*

Considering Jones' lemma (Jones, 1937), here is another case where space $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ is not normal. It is, in fact, a generalization for the previous theorem.

Theorem 2.6: Let A and B be subsets of $\mathbb{R}$. *If* $\mathbb{R}\backslash A$ and $\mathbb{R}\backslash B$ both contain an interval, then $(\mathbb{R}, \tau(A)) \times (\mathbb{R}, \tau(B))$ cannot be normal.

Recall that a space X is said to be α*-normal* if it is T_1 and for any pair of disjoint closed subsets, A and B of X there exist disjoint open subsets U and V such that $A \cap U$ is dense in A and $B \cap V$ is dense in B, (Ludwig, 2002). Clearly, T_4 implies α-normality. In Ludwig (2002), it was stated that "If X is an α-

normal space, then any two disjoint closed discrete subsets of X can be separated by two disjoint open subsets of X". Hence, we have the following statement.

Theorem 2.7: *Let A and B be subsets of $\mathbb{R}$. If $\mathbb{R}\backslash A$ and $\mathbb{R}\backslash B$ both contain an interval, then $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ cannot be α-normal.*

Proof: Assume, on the contrary, that $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is α-normal, where $\mathbb{R}\backslash A$ and $\mathbb{R}\backslash B$ contain an interval. Let $(a,b)\subseteq\mathbb{R}\backslash A$ and $(c,d)\subseteq\mathbb{R}\backslash B$. Let C be any line segment in $(a,b)\times(c,d)$ with a negative slope. By Jones' Lemma, pick two disjoint subsets E and F of C that cannot be separated by two disjoint open sets. Since C is closed and discrete in $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$, then so are E and F. This contradicts the assumption. Thus, $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is not α-normal.

Theorem 2.8: *If $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is Lindelöf or paracompact, then it is normal.*

Furthermore, since $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is always separable, we get the following result.

Theorem 2.9: *For any subsets, A and B of $\mathbb{R}$, $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is paracompact if and only if it is Lindelöf.*

The question *"under what condition is the product of two normal spaces normal?"* has been studied by many topologists. Partial answers to this question have been found. We apply some of those results to give explore when space $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is normal. By Theorem 2.1, we know that if A and B are subsets of $\mathbb{R}$ with countable complements, then $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is paracompact (and normal). However, observing that the product of a metrizable space and a perfectly normal space is paracompact and perfectly normal (Michael (1953), Proposition 5), we actually have the following "strengthening" of this statement.

Theorem 2.10*: If A and B are subsets of $\mathbb{R}$ with $\mathbb{R}\backslash A$ countable, then $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is perfectly normal and paracompact.*

Morita (1963) proved that "the product of a paracompact space which is a countable union of locally compact closed subsets, and a paracompact space is paracompact." Together with the paracompactness of every H-space we may deduce the following result.

Theorem 2.11: *If A and B are non-empty proper subsets of $\mathbb{R}$ such that $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is not normal, then both $(\mathbb{R},\tau(A))$ and $(\mathbb{R},\tau(B))$ are not a countable union of locally compact closed subsets.*

On the other hand, in Bouziad and Sukhacheva (2017), it was proved that *"The H-space $(\mathbb{R},\tau(A))$ is locally compact if and only if $\mathbb{R}\backslash A$ is closed in $(\mathbb{R},U)$ and discrete in $(\mathbb{R},S)$."* Thus, we get the following results.

Theorem 2.12: *If A and B are non-empty proper subsets of $\mathbb{R}$ such that $\mathbb{R}\backslash A$ is closed in $(\mathbb{R},U)$ and discrete in $(\mathbb{R},S)$, then $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is paracompact (and normal).*

Observe that, so far, all the normal products $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ which we have presented happen to be paracompact (Lindelöf). The question of whether there is a choice for subsets A and B of $\mathbb{R}$ such that $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is normal but not paracompact (Lindelöf) is still open.

Recall that a topological space X is called *L-normal* if there exists a normal space Y and a bijective function $f:X\longrightarrow Y$ such that the restriction $f\restriction_A:A\longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A\subseteq X$ (Kalantan and Saeed, 2017). In [12, Theorem 1.6], it was proved that "*If X is T_3 separable L-normal and of countable tightness, then X is T_4.*". Note that $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is of countable tightness, T_3 and separable. Hence, we have the following theorem.

Theorem 2.13: *For any subsets, A and B of $\mathbb{R}$, $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is L-normal if and only if $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is T_4.*

Recall that a topological space X is called *S-normal* if there exists a normal space Y and a bijective function $f:X\longrightarrow Y$ such that the restriction $f\restriction_A:A\longrightarrow f(A)$ is a homeomorphism for each separable subspace $A\subseteq X$ (Kalantan and Alhomieyed, 2018). Since $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is separable, the following is clear.

Theorem 2.14: *For any subsets, A and B of $\mathbb{R}$, $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is S-normal if and only if $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is T_4.*

Recall that a topological space X is called *C-normal* if there exists a normal space Y and a bijective function $f:X\longrightarrow Y$ such that the restriction $f\restriction_A:A\longrightarrow f(A)$ is a homeomorphism for each compact subspace $A\subseteq X$, (AlZahrani and Kalantan, 2017). If Y is paracompact, X is called *C_2-paracompact*, (Mohammed et al., 2019). For any subsets A and B of $\mathbb{R}$, we always have $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is submetrizable. Since every submetrizable space is C_2-paracompact (Mohammed et al., 2019), we deduce the following result.

Theorem 2.15: *For any subsets A and B of $\mathbb{R}$, the spaces $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is C-normal and C_2-paracompact.*

Recall that a topological space X is called *CC-normal,* (Kalantan and Alhomieyed, 2017), if there exists a normal space Y and a bijective function $f:X\longrightarrow Y$ such that the restriction $f\restriction_A:A\longrightarrow f(A)$ is a homeomorphism for each countably compact subspace $A\subseteq X$. We have the following theorem.

Theorem 2.16: *Let τ_1 and τ_2 be any topologies on $\mathbb{R}$, both finer than $\mathcal{U}$. Then, $(\mathbb{R},\tau_1)\times(\mathbb{R},\tau_2)$ is CC-normal.*

Proof: Consider $\mathbb{R}^2$ with its usual metric topology. Consider $id:(\mathbb{R},\tau_1)\times(\mathbb{R},\tau_2)\longrightarrow(\mathbb{R}^2,\mathcal{U})$. It is continuous because the usual metric is coarser than the product of τ_1 and τ_2. Let A be any countably compact subspace of $(\mathbb{R},\tau_1)\times(\mathbb{R},\tau_2)$. By continuity, A is countably compact in the metric space $(\mathbb{R}^2,\mathcal{U})$, hence A is closed in $(\mathbb{R}^2,\mathcal{U})$. For the same reason $id:(A,\tau_1\times\tau_2)\longrightarrow(A,\mathcal{U})$ is a homeomorphism as it is a bijection, continuous and closed: any closed subset E of A in $\tau_1\times\tau_2$ is countably compact (countable compactness is hereditary with respect to closed subspaces), thus $id(E)=E$ is countably compact in the metric space $(A,\mathcal{U})$, thus compact, hence closed.

Corollary 2.17: *For any A and B subsets of $\mathbb{R}$, $(\mathbb{R},\tau(A))\times(\mathbb{R},\tau(B))$ is CC-normal.*

Recall that a subset C of a topological space X is called a *closed domain* if $C=\overline{int(C)}$, i.e., C equals to the closure of its interior. We denote the family of all closed domains in X by $R[X]$. A *κ-metric* on a T_3 space is a non-negative real-valued function $\phi(x,C)$ of two variables, $x\in X$ and $C\in R[X]$, with the requirements:

- For every $x\in X$ and $C\in R[X]$, $\phi(x,C)=0\Leftrightarrow x\in C$.
- If $C,C'\in R[X]$ and $C\subset C'$, then $\phi(x,C)\geq\phi(x,C')$, for all $x\in X$.
- For every $C\in R[x]$, $\phi(x,C)$ is continuous in x.
- For every increasing transfinite sequence $\{C_\alpha\in R[X]:\alpha\in\Lambda\}$, we have:

$$\phi\left(x,\overline{\bigcup_{\alpha\in\Lambda}C_\alpha}\right)=\inf\{\phi(x,C_\alpha):\alpha\in\Lambda\}$$

Space on which there exists a κ-metric on it is said to be κ-metrizable (Ščepin, 1980). The concept of κ-metrizability is a generalization of metrizability, in the sense that every metric induces a κ-metric, so every metrizable space is κ-metrizable. In particular, $(\mathbb{R},U)$ is κ-metrizable.

The Sorgenfrey line is not metrizable, but it is κ-metrizable. Ščepin (1980) defined a κ-metric on the Sorgenfrey line as follows: Start by defining a bounded distance $d(x,y)=\min\{1,|x-y|\}$ for each $x,y\in\mathbb{R}$, and for each $x\in\mathbb{R}$, $\emptyset\neq E\subseteq\mathbb{R}$, define $d(x,E)=\inf\{d(x,y):y\in E\}$, and for each $x\in\mathbb{R}$ let $d(x,\emptyset)=1$. Now, define $\phi:X\times R[X]\to\mathbb{R}$ by the relation $\phi(x,C)=d(x,C\cap[x,\infty))$. With this definition, ϕ is a κ- metric on $(\mathbb{R},S)$, see (Suzuki et al., 1989) for more details.

It is unknown whether the H-topology $\tau(A)$ on $\mathbb{R}$ is κ-metrizable. However, neither the usual metric nor ϕ as defined above can be used to define a κ-metric on $(\mathbb{R},\tau(A))$. The usual metric $m:(\mathbb{R},\tau(A))\times R[\mathbb{R}]\to[0,\infty)$ defined by $m(x,C)=d(x,C)$ fails to satisfy (K1), for a counter-example, we may take $A=(-\infty,0)$, choose $C=[1,2)$. In this case, $m(2,C)=0$ while $2\notin C$. While it is easy to check that $\phi:(\mathbb{R},\tau(A))\times R[\mathbb{R}]\to[0,\infty)$ as defined previously satisfies (K1), (K2) and (K4); continuity of ϕ cannot be guaranteed. Here is an example.

Example 2.18: *Let $A=(-\infty,4)$, choose $C=[1,2]$. Considering $(\mathbb{R},\tau(A))$, we have $int(C)=(1,2)$ and $\overline{(1,2)}=[1,2]=C$. Hence, $C\in R[X]$. Let $x=2$. We show ϕ is discontinuous at x.*

Assume, on the contrary, that ϕ is continuous at $x=2$. Observe that $\phi(2,C)=0$ because $2\in C$. Consider $[0,\epsilon)$ with $0<\epsilon<1$, an open neighborhood of $0=\phi(2)$. Then, by assumed continuity at 2, there exists $\delta>0$ such that for all $y\in(2-\delta,2+\delta)$, we have $\phi(y,C)\in[0,\epsilon)$. However, for each $y\in(2,2+\delta)$, we have $\phi(y,C)=d(y,C\cap[y,\infty))=d(y,[1,2]\cap[y,\infty)=d(y,\emptyset)=1\notin[0,\epsilon)$, and $(2,2+\delta)\subset(2-\delta,2+\delta)$, a contradiction. Thus, ϕ is discontinuous at $x=2$ and is not a κ-metric on $(\mathbb{R},\tau(A))$. $.1in.1in.$

An interesting property of κ-metrizable spaces is that any product of κ-metrizable spaces is *κ-normal*. That is, any two disjoint closed domains could be separated by disjoint open sets (Ščepin, 1980). It is clear that every T_4 space is κ-normal, so this is a weaker version of normality. Hence, we are interested in κ-metrizablity of the general H-space and κ-normality of the non-normal products of H-spaces.

3. Conclusion

We have presented some of the topological properties of the product of two H-spaces. But, these problems are still open:

1. Can the product of two H-spaces be normal yet not paracompact (equivalently, not Lindelöf)?
2. If the product of two H-spaces is not normal, is it k-normal?
3. If the product of two H-spaces is not normal, is it α-normal?

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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