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Some properties of integration of real-valued function over a fuzzy interval

M. A. Shakhatreh *, A. M. Al-Shorman

Department of Mathematics, Yarmouk University, Irbid, Jordan

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ABSTRACT

One of the most fundamental concepts in fuzzy set theory is the extension principle. It gives a generic way of dealing with fuzzy quantities by extending non-fuzzy mathematical concepts. There are a few examples, including the concept of fuzzy distance between fuzzy sets. The extension approach is then methodically applied to real algebra, with considerable development of fuzzy number operations. These operations are computationally appealing and generalized interval analysis. Although the set of real fuzzy numbers with extended addition or multiplication is no longer a group, it retains many structural qualities. The extension concept is demonstrated to be particularly beneficial for defining set-theoretic operations for higher fuzzy sets. We need some definitions related to our properties before we can create the properties of integration of a crisp real-valued function over a fuzzy interval. It is our goal in this article to develop and demonstrate certain characteristics of a real-valued function over a fuzzy interval in order to broaden the scope of the notion of integration of a real-valued function over a fuzzy interval. Some of these characteristics are linked to the operations of extended addition and extended subtraction, while others are not.

1. Introduction

The extension principle introduced by Zadeh (1975) is one of the most basic ideas of fuzzy set theory. It provides a general method for extending non-fuzzy mathematical concepts in order to deal with fuzzy quantities. Some illustrations are given including the notion of fuzzy distance between fuzzy sets. The extension principle is then systematically applied to real algebra: Operations on fuzzy numbers are extensively developed. These operations generalize interval analysis and are computationally attractive. Although the set of real fuzzy numbers equipped with an extended addition or multiplication is no longer a group, many structural properties are preserved. Lastly, the extension principle is shown to be very useful for defining set-theoretic operations for higher fuzzy sets, then before building the properties of integration of a crisp real-valued function over a fuzzy interval we need some definitions related to our properties. To characterize the mathematical model of fuzzy HIV dynamics, the authors presented the approximate technique known as VIM in Saleh et al. (2021). In the linear fuzzy HIV infection model, VIM is reformulated and studied in terms of the fuzzy domain. Because of existing patients with varying immunization levels, it appears that the fuzzy HIV model can be used to depict the vulnerability-resistant cell level and the viral stack. The method series solution obeys the properties in the form of the triangular fuzzy number, according to the VIM results. The authors of Jameel et al. (2021) utilized HAM as a method for solving the fuzzy Volterra integral equation of the second kind with separable kernels. With application to the fuzzy Volterra integral equation of the second kind, fuzzy set theory was used to offer a new formulation of HAM. To determine the best value for the convergence-control parameter, the convergence of this strategy was qualitatively discussed. The method accurately approximates both linear and nonlinear fuzzy Volterra integral equations of the second type, as shown by numerical results and graphs. The authors of Stefanini et al. (2020) gave new calculus results for fuzzy-valued functions on a single real variable. They employ the midpoint-radius representation of intervals in the real half-plane extensively and demonstrate its utility in fuzzy calculus. Partial ordering and features of monotonicity and convexity are described and examined in detail using the midpoint representation of fuzzy-valued functions.

* Corresponding Author.
Email Address: mali@yu.edu.jo (M. A. Shakhatreh)
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Corresponding author's ORCID profile:
https://orcid.org/0000-0002-1704-1234

Based on the form of decomposition theorem, Wu (2019) suggested a new methodology for fuzzifying real-valued functions. They will also suggest that when fuzziness is taken into account, this new methodology prefers to be applied in real-world circumstances. The definition of new varieties of fuzzy number arithmetic is an interesting use of this new technology. This novel methodology can also be used to define the differentiations and integrals of fuzzy-number-valued functions. In Shakhatreh and Qawasmeh (2015), the authors introduced some concepts and definitions related to the Max-Min composition of fuzzy relations and prove the associativity of the Max-Min composition of three fuzzy relations. Shakhatreh and Qawasmeh (2020) proposed a new method to generate fuzzy equivalence relations in matrix form. They begin by constructing fuzzy equivalence relations in the matrix forms 3x3 and 4x4 matrices, and then use mathematical induction to construct a comprehensive approach for generating fuzzy equivalence relations of the form nxn matrices.

Definition 1 (Dubois, 1980; Zadeh, 1965): Let X be a universal set .Then a fuzzy set $\tilde{A}$ in X is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)): x \in X\}$, where $\mu_{\tilde{A}}(x)$ is the degree of membership of x in $\tilde{A}$ with $\mu_{\tilde{A}}(x) \in [0,1]$.

Definition 2 (Nguyen, 1978; Jain, 1976): (a) Let X be a Cartesian product of universal sets $X = X_1 \times X_2 \times \dots \times X_n$ and $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$ be n fuzzy sets in $X_1, X_2, \dots, X_n$, respectively, f is a mapping from X to a universal set Y, $y = f(x_1, x_2, \dots, x_n)$, and $f^{-1}(y) = \{x \in X: f(x) = y\}$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by:

$$\tilde{B} = f(\tilde{A}_1 \times \tilde{A}_2 \times \dots \times \tilde{A}_n) = \{(y, \mu_{\tilde{B}}(y)): y = f(x_1, x_2, \dots, x_n), (x_1, x_2, \dots, x_n) \in X\}$$

where,

$$\mu_{\tilde{B}}(y) = \begin{cases} sup_{(x_1, x_2, \dots, x_n) \in f^{-1}(y)} \ min\{\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2), \dots, \mu_{\tilde{A}_n}(x_n)\} & , f^{-1}(y) \neq \phi \\ 0 & , otherwise \end{cases}$$

(b) For $n = 1$, the extension principle reduces to:

$$\tilde{B} = f(\tilde{A}) = \{(y, \mu_{\tilde{B}}(y)): y = f(x), x \in X\}$$

where

$$\mu_{\tilde{B}}(y) = \mu_{f(\tilde{A})}(y) = \begin{cases} sup_{x \in f^{-1}(y)}\{\mu_{\tilde{A}}(x)\}, & f^{-1}(y) \neq 0 \\ 0 \quad , & otherwise \end{cases}$$

(c) If $n = 1$ and f is an injective function, then,

$$\mu_{\tilde{B}}(y) = \mu_{f(\tilde{A})}(y) = \begin{cases} \mu_{\tilde{A}}(f^{-1}(y)), f^{-1}(y) \neq 0 \\ 0 \quad , otherwise \end{cases}$$

Definition 3 (Zimmerman, 1996; Dubois and Prade, 1982): Let f be a crisp real-valued function that is integrable on the interval $I = [a_0, b_0]$. Then according to the extension principle, the membership function of the integral $\int_{\tilde{a}}^{\tilde{b}} f(u)du$ is given by:

$$\mu_{\int_{\tilde{a}}^{\tilde{b}} f(u)du}(z) = sup_{x,y \in I: \int_x^y f(u) = z} min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(x)\}.$$

If one of $\tilde{a}$ and $\tilde{b}$ is crisp (say $\tilde{a}$), then the integral of f over $[a, \tilde{b}]$ is given by:

$$\mu_{\int_a^{\tilde{b}} f(u)du}(z) = sup_{y: \int_a^y f(u)du} \mu_{\tilde{b}}(y) = sup_{y: F(y) - F(a) = z} \mu_{\tilde{b}}(y)$$

where F is an antiderivative of f.

Example 1: Let $\tilde{a} = \{(2,0.5), (3,0.8), (4,0.4)\}$, $\tilde{b} = \{(6,0.7), (7,0.9), (8,0.2)\}$, and $f(x) = 1$. Then $\int_a^b f(x)dx = b - a$.

According to definition 2, the detailed computational results of $\int_{\tilde{a}}^{\tilde{b}} f(x)dx$ are given in Table 1.

Hence choosing the maximum of the membership values for each value of the integral yields.

$$\int_{\tilde{a}}^{\tilde{b}} f(x)dx = \{(2,0.4), (3,0.7), (4,0.8), (5,0.5), (6,0.2)\}.$$

Definition 4 (Nahmias, 1978): A fuzzy set $\widetilde{M}$ is called a fuzzy number if the following condition holds:

a) $\widetilde{M}$ is a fuzzy set in the universe R
b) $\widetilde{M}$ is a convex fuzzy set
c) $\widetilde{M}$ is normalized fuzzy set
d) There exist a unique $X_0 \in R$ such that $\mu_{\widetilde{M}}(x_0) = 1$
e) $\mu_{\widetilde{M}}: R \rightarrow [0,1]$ is piecewise continuous

Table 1: Details computational results of $\int_{\tilde{a}}^{\tilde{b}} f(x)dx$

$[a,b]$	$\int_a^b f(x)dx$	$min\{\mu_{\tilde{a}}(a), \mu_{\tilde{b}}(b)\}$
[2,6]	4	0.5
[2,7]	5	0.5
[2,8]	6	0.2
[3,6]	3	0.7
[3,7]	4	0.8
[3,8]	5	0.2
[4,6]	2	0.4
[4,7]	3	0.4
[4,8]	4	0.2

Definition 5 (Zadeh, 1975; Kaufmann, 1975): Let $\tilde{A}$ and $\tilde{B}$ be two numbers. Then the extended addition defined by $\tilde{A} \oplus \tilde{B} = \{(z, \mu_{\tilde{A} \oplus \tilde{B}}(z)): z \in R\}$, where $\mu_{\tilde{A} \oplus \tilde{B}}(z) = sup_{x,y: x+y=z} min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$

Definition 6 (Nahmias, 1978): Let $\tilde{A}$ and $\tilde{B}$ be two numbers. Then the extended subtraction defined by $\tilde{A} \ominus \tilde{B} = \{(z, \mu_{\tilde{A}!\tilde{B}}(z)) : z \in R\}$, where $\mu_{\tilde{A}\ominus\tilde{B}}(z) = sup_{x,y:x-y=z} min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$

Now we can give and prove some properties of the integration of a real-valued function over a fuzzy interval according to definition 2.

Property 1: Let f be a function $f: I \to R$, integrable on I. Then $(\int_a^{\tilde{b}} f(u)du)^C \subseteq \int_a^{\tilde{b}^C} f(u)du$.
If $f: I \to R^+$ or $f: I \to R^-$, then the equality hold.

Proof:

$$\boldsymbol{\mu_{(\int_{\tilde{a}}^{\tilde{b}} f(u)du)}{}^{C}(z) = 1 - \mu_{\left(\int_{\tilde{a}}^{\tilde{b}} f(u)du\right)} = 1 - sup_{x:\int_a^x f(u)du} \mu_{\tilde{b}}(x) = inf_{x:\int_a^x f(u)du=z}\{1 - \mu_{\tilde{b}}(x)\} \leq sup_{x:\int_a^x f(u)du=z}\{\mu_{\tilde{b}^C}(x)\} = \mu_{\int_{\tilde{a}}^{\tilde{b}^C} f(u)du}(z) \Rightarrow \mu_{(\int_{\tilde{a}}^{\tilde{b}} f(u)du)}{}^{C}(z) \leq \mu_{\int_{\tilde{a}}^{\tilde{b}^C} f(u)du}(z)}$$

Therefore $(\int_a^{\tilde{b}} f(u)du)^C \subseteq \int_a^{\tilde{b}^C} f(u)du$.

Now, if $f: I \to R^+$ or $f: I \to R^-$, then

$$\boldsymbol{\mu_{(\int_{\tilde{a}}^{\tilde{b}} f(u)du)}{}^{C}(z) = 1 - \mu_{\left(\int_{\tilde{a}}^{\tilde{b}} f(u)du\right)}(z) = 1 - sup_{x:\int_a^x f(u)du} \mu_{\tilde{b}}(x).}$$

Since there exists a unique x_0 such that $\int_a^{x_0} f(u)du = z$,

$$\boldsymbol{\mu_{(\int_{\tilde{a}}^{\tilde{b}} f(u)du)}{}^{C}(z) = 1 - \mu_{\tilde{b}}(x_0) = \mu_{\tilde{b}^C}(x_0) = sup_{x:\int_a^x f(u)du=z}\{\mu_{\tilde{b}^C}(x)\} = \mu_{\int_a^{\tilde{b}^C} f(u)du}(z) \Rightarrow \mu_{(\int_{\tilde{a}}^{\tilde{b}} f(u)du)}{}^{C}(z) = \mu_{\int_a^{\tilde{b}^C} f(u)du}(z)}$$

Therefore $(\int_a^{\tilde{b}} f(u)du)^C = \int_a^{\tilde{b}^C} f(u)du$.

Property 2: Let f be a function $f: I \to R$, integrable on I. Then $(\int_{\tilde{a}}^{b} f(u)du)^C \subseteq \int_{\tilde{a}}^{b} f(u)du$.
If $f: I \to R^+$ or $f: I \to R^-$, then the equality hold.

Proof:

$$(\int_{\tilde{a}}^{b} f(u)du)^C = (-\int_b^{\tilde{a}} f(u)du)^C = (\int_b^{\tilde{a}} -f(u)du)^C \subseteq \int_b^{\tilde{a}^C} -f(u)du = -\int_b^{\tilde{a}^C} f(u)du = \int_{\tilde{a}^C}^{b} f(u)du$$

Property 3: Let f be a function $f: I \to R$, integrable on I.
If $\tilde{b} \subseteq \tilde{c}$, then $\int_a^{\tilde{b}} f(u)du \subseteq \int_a^{\tilde{c}} f(u)du$.

Proof:

$$\boldsymbol{\mu_{\int_a^{\tilde{b}} f(u)du}(z) = sup_{x:\int_a^x f(u)du=z} \mu_{\tilde{b}}(x) \leq sup_{x:\int_a^x f(u)du=z} \mu_{\tilde{c}}(x) = \mu_{\int_a^{\tilde{c}} f(u)du}(z) \Rightarrow \mu_{\int_a^{\tilde{b}} f(u)du}(z) \leq \mu_{\int_a^{\tilde{c}} f(u)du}(z).}$$

therefore, $\int_a^{\tilde{b}} f(u)du \subseteq \int_a^{\tilde{c}} f(u)du$.

Property 4: Let f and g be two functions $f: I \to R$ and $g: I \to R$ integrable on I. Then

(a) $\int_{\tilde{a}}^{\tilde{b}} [f(u) + g(u)]du \subseteq \int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{a}}^{\tilde{b}} g(u)du$
(b) $\int_{\tilde{a}}^{\tilde{b}} [f(u) - g(u)]du \subseteq \int_{\tilde{a}}^{\tilde{b}} f(u)du \,!\, \int_{\tilde{a}}^{\tilde{b}} g(u)du$

Proof (a):

$$\begin{aligned}
\boldsymbol{\mu_{\int_{\tilde{a}}^{\tilde{b}}[f(u)+g(u)]du}(z)} &= \boldsymbol{sup_{x,y:\int_x^y [f(u)+g(u)]du=z} min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}} \\
&= \boldsymbol{sup_{(x,y,x,y):\int_x^y f(u)du+\int_x^y g(u)du=z} min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y), \mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}} \\
&\leq \boldsymbol{sup_{(x_1,x_2,x_3,x_4):\int_{x_1}^{x_2} f(u)du+\int_{x_3}^{x_4} g(u)du=z} min\{\mu_{\tilde{a}}(x_1), \mu_{\tilde{b}}(x_2), \mu_{\tilde{a}}(x_3), \mu_{\tilde{b}}(x_4)\}} \\
&= \boldsymbol{sup_{(s,t):s+t=z,(x_1,x_2):\int_{x_1}^{x_2} f(u)du=s,(x_3,x_4):\int_{x_3}^{x_4} g(u)du=t} min\{\mu_{\tilde{a}}(x_1), \mu_{\tilde{b}}(x_2), \mu_{\tilde{a}}(x_3), \mu_{\tilde{b}}(x_4)\}} \\
&= \boldsymbol{sup_{(s,t):s+t=z} sup_{(x_1,x_2):\int_{x_1}^{x_2} f(u)du=s,(x_3,x_4):\int_{x_3}^{x_4} g(u)du=t} min\left\{min\{\mu_{\tilde{a}}(x_1), \mu_{\tilde{b}}(x_2)\}, min\{\mu_{\tilde{a}}(x_3), \mu_{\tilde{b}}(x_4)\}\right\}} \\
&= \boldsymbol{sup_{(s,t):s+t=z} min\{sup_{(x_1,x_2):\int_{x_1}^{x_2} f(u)du=s} min\{\mu_{\tilde{a}}(x_1), \mu_{\tilde{b}}(x_2)\}, sup_{(x_3,x_4):\int_{x_3}^{x_4} g(u)du=t} min\{\mu_{\tilde{a}}(x_3), \mu_{\tilde{b}}(x_4)\}\}} \\
&= \boldsymbol{sup_{(s,t):s+t=z} min\left\{\mu_{\int_{\tilde{a}}^{\tilde{b}} f(u)du}(s), \mu_{\int_{\tilde{a}}^{\tilde{b}} g(u)du}(t)\right\} = \mu_{\int_{\tilde{a}}^{\tilde{b}}[f(u)\oplus g(u)]du}(z) \Rightarrow \mu_{\int_{\tilde{a}}^{\tilde{b}}[f(u)+g(u)]du}(z)} \\
&\leq \boldsymbol{\mu_{\int_{\tilde{a}}^{\tilde{b}}[f(u)\oplus g(u)]du}(z)}
\end{aligned}$$

therefore,

$$\int_{\tilde{a}}^{\tilde{b}} [f(u) + g(u)]du \subseteq \int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{a}}^{\tilde{b}} g(u)du$$

Proof (b):

$$\int_{\tilde{a}}^{\tilde{b}} [f(u) - g(u)]du = \int_{\tilde{a}}^{\tilde{b}} [f(u) + (-g(u))]du \subseteq \int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{a}}^{\tilde{b}} -g(u)du = \int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \left(-\int_{\tilde{a}}^{\tilde{b}} g(u)du\right) = \int_{\tilde{a}}^{\tilde{b}} f(u)du \,!\, \int_{\tilde{a}}^{\tilde{b}} g(u)du$$

therefore,

$$\int_{\tilde{a}}^{\tilde{b}} [f(u) - g(u)]du \subseteq \int_{\tilde{a}}^{\tilde{b}} f(u)du \,!\, \int_{\tilde{a}}^{\tilde{b}} g(u)du.$$

Example: Let $f(x) = 2x - 2$, $g(x) = -2x + 3$, $\tilde{a} = \{(1,0.7), (2,1), (3,0.4)\}$, and $\tilde{b} = \{(3,0.6), (4,1), (5,0.3)\}$. Then $\int_a^b f(x)dx = [x^2 - 2x]_a^b$, $\int_a^b g(x)dx = [-x^2 + 3x]_a^b$, and $\int_a^b [f(x) + g(x)]dx = [x]_a^b$

According to definition 2, the detailed computational results of $\int_{\tilde{a}}^{\tilde{b}}[f(x)+g(x)]dx$ and $\int_{\tilde{a}}^{\tilde{b}} f(x)dx \oplus \int_{\tilde{a}}^{\tilde{b}} g(x)dx$ given in Table 2.

Table 2: Details computational results of $\int_{\tilde{a}}^{\tilde{b}}[f(x)+g(x)]dx$ and $\int_{\tilde{a}}^{\tilde{b}} f(x)dx \oplus \int_{\tilde{a}}^{\tilde{b}} g(x)dx$

$[a,b]$	$\boldsymbol{min\{\mu_{\tilde{a}}(a),\mu_{\tilde{b}}(b)\}}$	$\int_a^b f(x)dx$	$\int_a^b g(x)dx$	$\int_a^b [f(x)+g(x)]dx$
[1,3]	0.6	4	−2	2
[1,4]	0.7	9	−6	3
[1,5]	0.3	16	−12	4
[2,3]	0.6	3	−2	1
[2,4]	1	8	−6	2
[2,5]	0.3	15	−12	3
[3,3]	0.4	0	0	0
[3,4]	0.4	5	−4	1
[3,5]	0.3	12	−10	2

And hence,

$$\int_{\tilde{a}}^{\tilde{b}} f(x)dx = \{(0,0.4),(3,0.6),(4,0.6),(5,0.4),(8,1),(9,0.7),(12,0.3),(15,0.3),(16,0.3)\}$$

and

$$\int_{\tilde{a}}^{\tilde{b}} g(x)dx = \{(-12,0.3),(-10,0.3),(-6,1),(-4,0.4),(-2,0.6),(0,0.4)\} \Rightarrow$$

$$\int_{\tilde{a}}^{\tilde{b}}[f(x)+g(x)]dx = \{(0,0.4),(1,0.6),(2,1),(3,0.7),(4,0.3)\}$$

Applying the formula for the extended addition according to the extension principle yields.

$$\int_{\tilde{a}}^{\tilde{b}} f(x)dx \oplus \int_{\tilde{a}}^{\tilde{b}} g(x)dx = \{(12,0.3),(-10,0.3),(-9,0.3),(-8,0.3),(-7,0.3),(-6,0.4),(-5,0.3),(-4,0.4),(-3,0.6),(-1,0.4),(0,0.4),(1,0.6),(2,1),(3,0.7),(4,0.4),(5,0.4),(6,0.6),(7,0.6),(8,0.4),(9,0.5),(10,0.5),(11,0.4),(12,0.4),(13,0.5),(14,0.5),(15,0.4),(16,0.4)\}$$

Now, we can easily verify that:

$$\int_{\tilde{a}}^{\tilde{b}}[f(u)+g(u)]du \subseteq \int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{a}}^{\tilde{b}} g(u)du$$

Property 5: Let $f$be a function $f: I \to R$, integrable on I and $\tilde{b}$ be a normalized fuzzy set. Then

(a) $\int_{\tilde{a}}^{\tilde{c}} f(u)du \subseteq \int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{b}}^{\tilde{c}} g(u)du$

(b) If $\tilde{b}$ is a real number, then $\int_{\tilde{a}}^{\tilde{c}} f(u)du = \int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{b}}^{\tilde{c}} g(u)du$.

Proof (a):

$$\mu_{\int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{b}}^{\tilde{c}} f(u)du}(z) = sup_{(m,n):m+n=z} \min\left\{\mu_{\int_{\tilde{a}}^{\tilde{b}} f(u)du}(m), \mu_{\int_{\tilde{b}}^{\tilde{c}} f(u)du}(n)\right\} =$$
$$sup_{(m,n):m+n=z} \min\left\{sup_{(x_1,x_2):\int_{x_1}^{x_2} f(u)du=m} \min\{\mu_{\tilde{a}}(x_1),\mu_{\tilde{a}}(x_2)\}, sup_{(x_3,x_4):\int_{x_3}^{x_4} f(u)du=n} \min\{\mu_{\tilde{b}}(x_3),\mu_{\tilde{c}}(x_4)\}\right\} =$$
$$sup_{(m,n):m+n=z} sup_{(x_1,x_2):\int_{x_1}^{x_2} f(u)du=m} sup_{(x_3,x_4):\int_{x_3}^{x_4} f(u)du=n} \min\{\min\{\mu_{\tilde{a}}(x_1),\mu_{\tilde{a}}(x_2)\}, \min\{\mu_{\tilde{b}}(x_3),\mu_{\tilde{c}}(x_4)\}\} =$$
$$sup_{(m,n):m+n=z} sup_{(x_1,x_2):\int_{x_1}^{x_2} f(u)du=m} sup_{(x_3,x_4):\int_{x_3}^{x_4} f(u)du=n} \min\{\mu_{\tilde{a}}(x_1),\mu_{\tilde{a}}(x_2),\mu_{\tilde{b}}(x_3),\mu_{\tilde{c}}(x_4)\} =$$
$$sup_{(x_1,x_2,x_3,x_4)\int_{x_1}^{x_2} f(u)du+\int_{x_3}^{x_4} f(u)du} \min\{\mu_{\tilde{a}}(x_1),\mu_{\tilde{a}}(x_2),\mu_{\tilde{b}}(x_3),\mu_{\tilde{c}}(x_4)\} \geq$$
$$sup_{(x_1,x_2,x_3,x_4)\int_{x_1}^{x_2} f(u)du+\int_{x_3}^{x_4} f(u)du=z\,\&\,x_2=x_3:} \min\{\mu_{\tilde{a}}(x_1),\mu_{\tilde{a}}(x_2),\mu_{\tilde{b}}(x_3),\mu_{\tilde{c}}(x_4)\} =$$
$$sup_{(x_1,y,y,x_4)\int_{x_1}^{y} f(u)du+\int_{y}^{x_4} f(u)du=z:} \min\{\mu_{\tilde{a}}(x_1),\mu_{\tilde{b}}(y),\mu_{\tilde{c}}(x_4)\} = sup_{(x_1,y,y,x_4)\int_{x_1}^{x_4} f(u)du=z:} \min\{\mu_{\tilde{b}}(y),\min\{\mu_{\tilde{a}}(x_1),\mu_{\tilde{c}}(x_4)\}\}$$

Since $\tilde{b}$ is normalized set,

$$\mu_{\int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{b}}^{\tilde{c}} f(u)du}(z) \geq sup_{(x_1,x_4)\int_{x_1}^{x_4} f(u)du} min\{\mu_{\tilde{a}}(x_1),\mu_{\tilde{a}}(x_4)\} = \mu_{\int_{\tilde{a}}^{\tilde{c}} f(u)du}(z)$$

Hence $\mu_{\int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{b}}^{\tilde{c}} f(u)du}(z) \geq \mu_{\int_{\tilde{a}}^{\tilde{c}} f(u)du}(z)$

Therefore $\int_{\tilde{a}}^{\tilde{c}} f(u)du \subseteq \int_{\tilde{a}}^{\tilde{b}} f(u)du \oplus \int_{\tilde{b}}^{\tilde{c}} g(u)du$

Proof (b):

If $\tilde{b}$ is a real number, then

$$\mu_{\int_{\tilde{a}}^{\tilde{c}} f(u)du}(z) = sup_{(x,y):\int_x^y f(u)du=z} \min\{\mu_{\tilde{a}}(x),\mu_{\tilde{c}}(x)\}$$
$$= sup_{(x,y):\int_x^b f(u)du+\int_b^y f(u)du} \min\{\mu_{\tilde{a}}(x),\mu_{\tilde{c}}(x)\}$$
$$= \mu_{\int_{\tilde{a}}^{b} f(u)du \oplus \int_{b}^{\tilde{c}} f(u)du}(z) \Rightarrow \mu_{\int_{\tilde{a}}^{\tilde{c}} f(u)du}(z)$$
$$= \mu_{\int_{\tilde{a}}^{b} f(u)du \oplus \int_{b}^{\tilde{c}} f(u)du}(z)$$

Therefore $\int_{\tilde{a}}^{\tilde{c}} f(u)du = \int_{\tilde{a}}^{b} f(u)du \oplus \int_{b}^{\tilde{c}} g(u)du$

2. Conclusion

The extension principle is one of the most essential principles in fuzzy set theory. By extending non-fuzzy mathematical notions, it provides a generic technique to deal with fuzzy quantities. There are a few examples, such as the fuzzy distance between fuzzy sets idea. After that, the extension method is applied to real algebra, with a focus on fuzzy number operations. These methods, as well as generalized interval analysis, are computationally interesting. Although the set of real fuzzy numbers with extended addition or multiplication is no longer a group, it nevertheless has a number of structural characteristics. It is shown that the extension notion is particularly useful for developing set-theoretic procedures for higher fuzzy sets. Before we can build the properties of integration of a crisp real-valued function over a fuzzy interval, we need some definitions relevant to our properties. In this paper, we develop and prove some real-valued function properties over a fuzzy interval to extend the principle of real-valued function integration over a fuzzy interval. Such features are correlated with extended addition and extended subtraction.

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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