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Analysis of Gegenbauer kernel filtration on the hypersphere

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ABSTRACT

In this study, we aim to construct explicit forms of convolution formulae for Gegenbauer kernel filtration on the surface of the unit hypersphere. Using the properties of Gegenbauer polynomials, we reformulated Gegenbauer filtration as the limit of a sequence of finite linear combinations of hyperspherical Legendre harmonics and gave proof for the completeness of the associated series. We also proved the existence of a fundamental solution of the spherical Laplace-Beltrami operator on the hypersphere using the filtration kernel. An application of the filtration on a one-dimensional Cauchy wave problem was also demonstrated.

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1. Introduction

Spherical harmonic analysis is basically the spectral theory of a differential operator such as the spherical Laplacian, which we denote by Δ_{S^n}, on the hypersphere. In this analysis, spherical harmonics play salient roles. Spherical harmonic analysis is a process of decomposing a function on a sphere into components of various wavelengths using surface spherical harmonics as base functions (Bogdanova et al., 2005; Bulow, 2004).

The role of classical orthogonal polynomials such as the Gegenbauer polynomials as reproducing kernels for the spaces of spherical harmonics of a given degree, or more generally, as providing an explicit construction of symmetry adapted basis functions for those spaces has been studied extensively (Camporesi, 1990; Bezubik and Strasburger, 2006; Omenyi and Uchenna, 2019). Also, Strasburger (1993) studied the connection of the Fourier transform on the Euclidean space to the Hankel transform obtained via a restriction to $SO(n)$-finite functions and various integral identities of the Hecke-Bochner type resulting there. The generalized concept of convolution on groups is intimately related to the concept of filtering on homogeneous spaces. Some insight into spherical filtering with particular emphasis on wavelet transform can be found in Driscoll and Healy (1994), Antoine and Vandergheynst (1999), Bogdanova et al. (2005), and more recently in Dai and Xu (2013) and Claessens (2016).

The goal of the present paper is to present a novel form of the Gegenbauer kernel filtration of harmonic functions on the hypersphere. It is known, in general, that there is no explicit expression for the fundamental solution of a Laplace-type operator on a Riemannian manifold (Aubin, 1998; Cohl and Palmer, 2015). In this work, we aim to demonstrate that with the Gegenbauer filtration kernel, a closed-form of fundamental solution can be constructed. This puts in limelight signal processing methods on non-Euclidean spaces and in particular on the hypersphere. The most basic is the notion of Fourier transform that on the sphere corresponds to the expansion of functions into series of familiar spherical harmonics. A vast amount of literature is available on such expansions, mostly from quantum mechanics and mathematical physics (Szekeres, 2004; Assche et al., 2000; Cohl and Palmer, 2015; Drake et al., 2008; Healy et al., 2003). We also derive some general formulae for the Gegenbauer filtration of functions on S^n, including recent generalizations of Fourier spherical harmonic expansions and discuss their function theoretic consequences.

In the next subsections, we fix basic notations and concepts before proceeding to present our results in subsequent sections.

1.1. The hypersphere

The hypersphere, $S^{n-1} = \{x \in \mathbb{R}^n : x^T x = 1\}$, $n > 3$ in $\mathbb{R}^n$ is a set of points whose Euclidean distance from the origin is equal to unity. The hypersphere S^{n-1} may be parameterized by a set of

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hyperspherical polar coordinates. If $(x_1, x_2, \cdots, x_n)$ are Cartesian coordinates in $\mathbb{R}^n$, then we define the angles $\theta_1, \theta_2, \cdots \theta_{n-1}$ with $\theta_1, \theta_2, \cdots, \theta_{n-2} \in [0, \pi]$ and $\theta_{n-1} \in [0, 2\pi]$ such that:

$$\left.\begin{array}{rcl} x_1 & = & \cos\theta_1 \\ x_2 & = & \sin\theta_1 \cos\theta_2 \\ x_3 & = & \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ x_4 & = & \sin\theta_1 \sin\theta_2 \sin\theta_3 \\ \vdots & & \\ x_{n-1} & = & \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \cos\theta_{n-1} \\ x_n & = & \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \sin\theta_{n-1}. \end{array}\right\} \quad (1)$$

This is a natural generalization of spherical polar coordinates in $\mathbb{R}^3$. In the familiar case of $S^2 \subset \mathbb{R}^3$, θ_1 corresponds to the elevation and θ_2 corresponds to the azimuth.

The surface area of the hypersphere satisfies the recursive relation:

$$|S^n| = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}, \; n \geq 3, \quad (2)$$

see e.g. Jost and Jost (2008) and Lee (2003) for details.

Let $\mathcal{H}_{l,n}$ denote the space homogeneous Legendre polynomials of degree l in dimension n. We call function $f \in \mathcal{H}_{l,n}$ such that $\Delta_{S^n} f = 0$ a hyperspherical harmonic, where Δ_{S^n} is the spherical Laplacian defined as:

$$\Delta_{S^n} := \frac{1}{\sin^{n-1}\theta} \frac{\partial}{\partial\theta} (\sin^{n-1} \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta} \Delta_{S^{n-1}}. \quad (3)$$

The space of hyperspherical harmonic polynomials restricted to the unit hypersphere, S^n, is denoted by $\mathcal{Y}_{l,n}$. So, any $Y_l \in \mathcal{Y}_{l,n}$ is related to a homogeneous harmonic $h_l \in \mathcal{H}_{l,n}$ by $h_l(r\xi) = r^l Y_l(\xi)$ where $r = |h_l|$. So, they have the same dimension.

1.2. Hyperspherical harmonics

To construct a Gegenbauer kernel for filtration on the hypersphere, one needs to clarify the concept of Gegenbauer polynomials, which we denote by C_l^α for degree l and index α, expressible through hyperspherical Legendre polynomials $\{P_{l,n}(x): l = 0,1,2,\cdots\}$.

Definition 1.1: The function $P_{l,n}$ is hyperspherical Legendre polynomial of degree l in n dimension and it is given by:

$$P_{l,n}(t) := l!\,\Gamma(\tfrac{n-1}{2}) \sum_{j=0}^{[\frac{l}{2}]} (-1)^j \frac{(1-t^2)^j t^{l-2j}}{4^j j!(l-2j)!\Gamma(j+\frac{n-1}{2})}, \; t \in [-1,1]; \quad (4)$$

where $[x]$ is the smallest integer greater or equal to $x \in \mathbb{R}$. It is orthogonal with respect to the weight function $(1-x^2)^{\alpha-\frac{1}{2}}$ on the support interval $[-1,1]$. The Rodrigues representation of $P_{l,n}$ is:

$$P_{l,n}(t) = (-1)^l R_{l,n} (1-t^2)^{\frac{3-n}{2}} \frac{d^l}{dt^l} (1-t^2)^{l+\frac{n-3}{2}} \text{ with } n \geq 2, \quad (5)$$

where the Rodrigues constant $R_{l,n}$ is given by:

$$R_{l,n} = \frac{\Gamma(\frac{n-1}{2})}{2^l \Gamma(l + \frac{n-1}{2})}.$$

We remark that for $n = 3$, one recovers from 5 the standard Rodrigues representation formula for the standard Legendre polynomials as:

$$P_{l,3}(t) = \frac{1}{2^l l!} (\frac{d}{dt})^l (t^2-1)^l, \; l \in \mathbb{N}_0. \quad (6)$$

Moreover, Morimoto (1998) proved an integral representation of $P_{l,n}$ to be:

$$P_{l,n}(t) = \frac{|S^{n-3}|}{|S^{n-2}|} \int_{-1}^{1} [t + i(1-t^2)^{\frac{1}{2}} s]^l (1-s^2)^{\frac{n-4}{2}} ds; \; l \in \mathbb{N}_0, n \geq 3, t \in [-1,1]. \quad (7)$$

We recall that $P_{l,n}(t) \in \mathcal{H}_l(S^n)$ and note that the dimension, $d_l(n)$, of $\mathcal{H}_l(S^n)$ is given by the formula:

$$d_l(n) = \binom{l+n}{n} - \binom{l+n-2}{n} = \frac{(2l+n-1)(l+n-2)!}{l!\,(n-1)!}, \quad \text{for } l \in \mathbb{N}.$$

with index α See e.g. (Avery, 2012; Atkinson and Han, 2012; Wong, 2006; Omenyi, 2014; Omenyi and Uchenna, 2019; Morimoto, 1998) for details. A generalization of $P_{l,n}$ is the l degree Gegenbauer polynomial, C_l^α, in index α defined by:

$$C_l^\alpha(t) := \binom{l+2\alpha-1}{l} \frac{\Gamma(\alpha+1/2)}{\sqrt{\pi}\Gamma(\alpha)} \int_{-1}^{1} (t + i(1-t^2)^{1/2} s)^l (1-s^2)^{\alpha-1} ds, \alpha > 0, l \in \mathbb{N}_0. \quad (8)$$

2. Technical lemmas and basic assumptions

In what follows, we briefly review basic technical lemmas and assumptions on hyperspherical harmonic polynomials that will lead us to the main result of this work.

Lemma 2.1: (Addition lemma). Let $\{\psi_{l,j}: 1 \leq j \leq d_l(n)\}$ be an orthonormal basis of $\mathcal{H}_l(S^n)$, i.e:

$$\int_{S^n} \psi_{l,j}(x) \bar{\psi}_{l,m}(x) dV_g(x) = \delta_{jm}; \;\; 1 \leq j, m \leq d_l(n). \quad (9)$$

Then,

$$\sum_{j=1}^{d_l(n)} \psi_{l,j}(x) \bar{\psi}_{l,m}(y) = \frac{d_l(n)}{|S^n|} P_{l,\frac{(n-1)}{2}}(x \cdot y). \quad (10)$$

For proof, one may see Omenyi and Uchenna (2019), Omenyi (2014), and Morimoto (1998). This means in particular that $P_{l,(n-1)/2}(x \cdot y)$ is a harmonic function on S^n with eigenvalue $\lambda_l = l(l+n-1)$ for the eigenvalue problem:

$$\Delta_{S^n} P_{l,(n-1)/2}(\theta) = \lambda_l P_{l,(n-1)/2}(\theta).$$

Lemma 2.2 (Morimoto, 1998): The hyperspherical harmonic polynomials are orthogonal:

$$\int_{S^n} P_{l,n}(\xi\cdot\eta)P_{j,n}(\xi\cdot\eta)dV_n(\xi)=\begin{cases}\frac{|S^n|}{d_l(n)} & \text{if } l=j.\\ 0 & \text{if } l\neq j.\end{cases} \quad (11)$$

An interesting assumption comes from the projection of integrable function onto the space of spherical harmonics on the hypersphere. We make the following definition.

Definition 2.3: A projection $F_{l,n}$ of $f\in L^1(S^{n-1})$ into $\mathcal{Y}_{l,n}$ is defined to be:

$$(F_{l,n}f)(\xi):=\frac{d_l(n)}{|S^{n-1}|}\int_{S^{n-1}} P_{l,n}(\xi\cdot\eta)f(\eta)dS_{n-1}(\eta),\ \eta\in S^{n-1}.$$
(12)

Assumption 2.4: If $f\in L^2(S^{n-1})$ then for any $\xi\in S^{n-1}$ we have:

$$||(F_{l,n}f)||_{L^2(S^{n-1})}\leq ||f||_{L^2(S^{n-1})}.$$

Proof: Let $f\in L^2(S^{n-1})$ and $\xi\in S^{n-1}$ given, we have:

$$|(F_{l,n}f)(\xi)|^2\leq\frac{d_l(n)}{|S^{n-1}|}\int_{S^{n-1}}|P_{l,n}(\xi\cdot\eta)|^2dS_{n-1}(\eta)\cdot\int_{S^{n-1}}|f(\eta)|^2dS_{n-1}(\eta)$$
$$||(F_{l,n}f)||_{L^2(S^{n-1})}\leq\sqrt{d_l(n)}||f||_{L^2(S^{n-1})}.$$

Similarly,

$$||(F_{l,n}f)||_{L^2(S^{n-1})}\leq\sqrt{\frac{d_l(n)}{|S^{n-1}|}}||f||_{L^2(S^{n-1})}.$$

These imply that $||(F_{l,n}f)||_{L^2(S^{n-1})}\leq ||f||_{L^2(S^{n-1})}$. The orthogonal decomposition of the hyperspherical harmonics 9 and the addition lemma 2.1 imply that any $f\in L^2(S^{n-1})$ can be uniquely represented as:

$$f(\xi)=\sum_{l=0}^{\infty}f_l(\xi);\ \text{ with } f_l\in\mathcal{Y}_{l,n};\ l\geq 0. \quad (13)$$

We call $f_l\in\mathcal{Y}_{l,n}$ hyperspherical component of f given by:

$$f_l(\xi)=\frac{d_l(n)}{|S^{n-1}|}\int_{S^{n-1}} f(\eta)P_{l,n}(\xi\cdot\eta)dS_{n-1}(\eta);\quad \eta\geq 0. \quad (14)$$

Lemma 2.5 (Morimoto, 1998): The Gegenbauer function C_l^{α} is indeed a polynomial and has a representation in terms of the hyperspherical harmonics as:

$$C_l^{\frac{n-2}{2}}(t):=\binom{l+n-3}{l}P_{l,n}(t);\quad\text{for}\quad n\geq 3. \quad (15)$$

3. Results and discussion

We now present the main results of this study. Let $f\in C(S^{n-1})$. We define:

$$f(\xi):=\int_{S^{n-1}}\delta(1-\xi\cdot\eta)f(\eta)dS_{n-1}(\eta),\qquad \xi\in S^{n-1}$$

using a Dirac delta function $\delta(t)$ whose value is defined as:

$$\delta(t):=\begin{cases}0 & \text{if } t\neq 0\\ +\infty & \text{if } t=0.\end{cases}$$

and satisfies

$$\int_{S^{n-1}}\delta(1-\xi\cdot\eta)dS_{n-1}(\eta)=1\ \ \forall\xi\in S^{n-1}.$$

One can construct a sequence of kernel functions $G_l(t)$ such that $G_l(\xi\cdot\eta)$ approaches $\delta(1-\xi\cdot\eta)$ and is such that for each $l\in\mathbb{N}$, the function

$$\int_{S^{n-1}}G_l(\xi\cdot\eta)dS_{n-1}(\eta)$$

is a linear combination of spherical harmonics of the order less than or equal to l. One possibility is to choose $G_l(t)$ proportional to $\frac{(1+t)^l}{2^l}$. Thus, we let:

$$G_l(t)=\alpha_{l,n}(\frac{1+t}{2})^l$$

where $\alpha_{l,n}$ is a scaling constant so that:

$$\int_{S^{n-1}}G_l(\xi\cdot\eta)dS_{n-1}(\eta)=1\ \ \forall\xi\in S^{n-1}. \quad (18)$$

Proposition 3.1: The constant $\alpha_{l,n}$ is given by:

$$\alpha_{l,n}=\frac{(l+n-2)!}{(4\pi)^{\frac{n-1}{2}}\Gamma(l+\frac{n-1}{2})}. \quad (19)$$

Proof. From $\int_{S^{n-1}}(\frac{1+t}{2})^l dS_{n-1}(\eta)=|S^{n-1}|\int_{-1}^{1}(\frac{1+t}{2})^l(1-t^2)^{\frac{n-3}{2}}dt$ we deduce on change of variables: $s=\frac{1+t}{2}$ that:

$$\int_{S^{n-1}}(\frac{1+t}{2})^l dS_{n-1}(\eta)=2^{n-2}|S^{n-2}|\int_0^1 s^{l+\frac{n-3}{2}}ds$$

and from 2 we know that $|S^{n-2}|=\frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$. Moreover,

$$\int_0^1 s^{l+\frac{n-3}{2}}(1-s)^{\frac{n3}{2}}ds=\beta(l+\frac{n-1}{2},\frac{n-1}{2})$$
$$=\frac{\Gamma(l+\frac{n-1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(l+n-1)}.$$

Thus,

$$\int_{S^{n-1}}(\frac{1+\xi\cdot\eta}{2})^l dS_{n-1}(\eta)=(4\pi)^{\frac{n-1}{2}}\frac{\Gamma(l+\frac{n-1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(l+n-1)}.$$

Therefore, $\alpha_{l,n}$ has the proposed value.

Now we introduce a filtration operator $\mathcal{G}_{l,n}$ defined as follows:

$$(\mathcal{G}_{l,n}f)(\xi):=\alpha_{l,n}\int_{S^{n-1}}(\frac{1+\xi\cdot\eta}{2})^l f(\eta)dS_{n-1}(\eta),\quad\forall f\in C(S^{n-1}).$$

We express $(\mathcal{G}_{l,n}f)(\xi)$ as a linear combination of spherical harmonics of the order less than or equal to l. To do this, write:

$$\alpha_{l,n}(\frac{1+t}{2})^l = \sum_{k=0}^{l} \mu_{l,k,n} \frac{d_l(n)}{|S^{n-1}|} P_{l,n}(t) \quad (20)$$

and thus from 15:

$$\alpha_{l,n}(\frac{1+t}{2})^l = \sum_{k=0}^{l} \mu_{l,k,n} C_l^{\frac{n-2}{2}}(t). \quad (21)$$

To determine the coefficients $\{\mu_{l,k,n}\}_{k=0}^{l}$, we multiply both sides by the function:

$$P_{j,n}(t)(1-t^2)^{\frac{n-3}{2}},\ 0 \le j \le l,$$

integrate from $t=-1$ to $t=1$ and use the orthogonality condition of $P_{l,n}$ to obtain:

$$\mu_{l,k,n} = \alpha_{l,n}|S^{n-2}| \int_{-1}^{1} (\frac{1+t}{2})^l P_{l,n}(t)(1-t^2)^{\frac{n-3}{2}} dt.$$

Computing this following the same procedure as in the derivation of $\alpha_{l,n}$ we get

$$\mu_{l,k,n} = \frac{l!\,(l+n-2)!}{(l-j)!\,(l+j+n-2)!}.$$

So we have the definition of Gegenbauer filtering given by

$$(\mathcal{G}_{l,n}f)(\xi) = \sum_{k=0}^{l} \mu_{l,k,n}(F_{l,n}f)(\xi) \quad (22)$$

where F is the projector defined in 12. In order words, $\mathcal{G}_{l,n}$ is a linear combination of spherical harmonics of the order less or equal to l. We also observe that for $t \in [-1,1)$,:

$$\lim_{l\to\infty}(\alpha_{l,n}(\frac{1+t}{2})^l) = 0.$$

Now we state and prove one of the main results of this study.

Theorem 3.2: The Gegenbauer filtration operator $\mathcal{G}_{l,n}$ is complete. That is, let $f \in C(S^{n-1})$ then $lim_{l\to\infty}||\mathcal{G}_{l,n}f - f||_{C(S^{n-1})} = 0$.

Proof. Using modulus of continuity, we have:

$$\omega(f;\delta) := \sup\{|f(\xi)-f(\eta)| : \xi,\eta \in S^{n-1}, |\xi-\eta| \le \delta\},\quad \delta > 0,$$

and since $f \in C(S^{n-1})$, we have that $\omega(f;\delta) \to 0$ as $\delta \to 0$. Denote:

$$M := \sup\{|f(\xi)-f(\eta)| : \xi,\eta \in S^{n-1}, |\xi-\eta| \le \delta\} < \infty.$$

Let $\xi \in S^{n-1}$ be arbitrary but fixed. Using (18), we have:

$$(\mathcal{G}_{l,n}f)(\xi) - f(\xi) = \alpha_{l,n}\int_{S^{n-1}} (\frac{1+\xi\cdot\eta}{2})^l [f(\eta) - f(\xi)] dS_{n-1}(\eta) = I_1(\xi) + I_2(\xi),$$

where,

$$I_1(\xi) = \alpha_{l,n}\int_{\eta\in S^{n-1}:|\xi-\eta|\le\delta} (\frac{1+\xi\cdot\eta}{2})^l [f(\eta) - f(\xi)] dS_{n-1}(\eta)$$

and,

$$I_2(\xi) = \alpha_{l,n}\int_{\eta\in S^{n-1}:|\xi-\eta|>\delta} (\frac{1+\xi\cdot\eta}{2})^l [f(\eta) - f(\xi)] dS_{n-1}(\eta).$$

we bound each term as follows.

$$I_1(\xi) \le \omega(f;\delta)\alpha_{l,n}\int_{S^{n-1}} (\frac{1+\xi\cdot\eta}{2})^l dS_{n-1}(\eta) = \omega(f;\delta)$$

and,

$$I_2(\xi) \le M\alpha_{l,n}|S^{n-1}|(1-\frac{\delta^2}{2})^l.$$

In bounding $I_2(\xi)$, we used the relation

$$|\xi-\eta| > \delta\xi\cdot\eta < 1-\frac{\delta^2}{2}$$

for $\xi,\eta \in S^{n-1}$. Thus, for any $\delta \in (0,1)$, applying Theorem 3.2, we have:

$$\limsup_{l\to\infty}||\mathcal{G}_{l,n}f - f||_{C(S^{n-1})} \le \omega(f;\delta).$$

note that $\omega(f;\delta) \to 0$ as $\delta \to 0$. So the statement holds.

Using the operator 22, we can recast Theorem 3.2 as 3.3 below.

Theorem 3.3: *For any* $f \in C(S^{n-1})$,

$$f(\xi) = \limsup_{l\to\infty}\sum_{k=0}^{l} \mu_{l,k,n}(\mathcal{G}_{k,n}f)(\xi)$$

uniformly in $\xi \in S^{n-1}$. If $\mathcal{G}_{k,n}f = 0$ for all $l \in \mathbb{N}_0$, then f must be the zero function.

Theorem 3.3 and the orthogonality of the hyperspherical polynomials imply that $\{\mathcal{Y}_l^n : l \in \mathbb{N}_0\}$ is the only system of source spaces in $C(S^{n-1})$ since any primitive space not identical to one in $\mathcal{Y}_l^n$, $l \in \mathbb{N}_0$ is orthogonal to all and is therefore trivial.

The inner product of two complex-valued functions on the surface of S^{n-1} is:

$$\langle f,h\rangle_{S^{n-1}} = \int_{S^{n-1}} f(\xi,\eta)h(\xi,\eta)^* dS_{n-1}(\eta)$$

where $*$ denotes complex conjugation. Using the fact that the Gegenbauer filtration is rotation invariant over $SO(n)$, one can move h to any point $(\xi_0,\eta_0) \in S^{n-1}$. Then we define a generalized convolution as a function on the rotation group $SO(n)$ to be:

$$(h*f)(R) = \int_{S^{n-1}} f(\xi,\eta)h_R(\xi,\eta)^* dS_{n-1}(\eta)$$

for $R \in SO(n)$ and where h_R is h rotated by R defined as $h_R(A) = h(R^{-1}A)$. Thus, every well-behaved function $f \in L^2(S^{n-1})$ admits the expansion

$$f(\xi,\eta) = \sum_{\alpha=0}^{\infty} \sum_{l,m} \hat{f}_{l,m}(\xi,\eta) C_l^{\alpha}(\xi,\eta)$$

where the generalized Fourier transform on the surface of the hypersphere $\hat{f}_{l,m}$ is defined to be:

$$\hat{f}_{l,m}(\xi) = \int_{S^{n-1} \ni \eta \in SO(n)} f(R)\rho_{l,m}(R) dS_{n-1}(\eta).$$

Here, $\rho(R)$ is a function on $SO(n)$ containing fixed matrix-valued functions called the irreducible representations of $SO(n)$.

As a consequence of 12, we observe that for a suitable ψ, we have:

$$(F_{l,n}\psi)(\xi) = \frac{d_l(n)}{|S^{n-1}|} \int_{S^{n-1}} P_{l,n}(\xi \cdot \eta)\psi(\eta) dS_{n-1}(\eta)$$

$$= \frac{d_l(n)}{|S^{n-1}|} \frac{|S^{n-1}|}{d_l(n)} \psi_l(\xi) = \sum_{j=1}^{d_l(n)} \langle \psi_l, \psi_{l,j} \rangle_{S^{n-1}} \psi_{l,j}(\xi)$$

$$\Rightarrow (F_{l,n}\psi)(\xi) = \sum_{j=1}^{d_l(n)} \hat{\psi}_l \psi_{l,j}(\xi). \qquad (23)$$

This leads to another interesting result of this work that Gegenbauer filtration on S^{n-1} coincides with convolution on S^{n-1}. Particularly, the convolution of two functions on the hypersphere equals the multiplication of their Fourier coefficients. This is the next Theorem 3.4.

Theorem 3.4: Let $f, h \in L^2(S^{n-1})$ and $\mathcal{G}_{l,n}$ the Gegenbauer filtration operator. Then the spectrum of the Gegenbauer kernel convolution is given by:

$$\widehat{f * h}(l,\alpha) = \sqrt{\frac{\Gamma(l+n-2)(2l+n-2)}{l!\Gamma(n-1)}} \hat{f}(l,\alpha)\hat{h}(l,0). \qquad (24)$$

A special case of this property has been proved for the special case of convolutions on S^2, see e.g. (Driscoll and Healy, 1994; Bulow, 2004; Bezubik and Strasburger, 2006). We get proof for this generalization on the hypersphere, S^{n-1}.

Proof. Since $f, h \in L^2(S^{n-1})$ and by definition of the Gegenbauer filtration,

$$\widehat{f * h}(l,\alpha) = \int_{S^{n-1}} f * h(\xi) \overline{\mathcal{G}_{l,n}(\xi)} dS_{n-1}(\xi)$$

$$= \int_{S^{n-1}} \left(\int_{SO(n)} f(R\eta) h(R^{-1}\xi) dR \right) \overline{\mathcal{G}_{l,n}(\xi)} dS_{n-1}(\xi).$$

After rearranging the integrals and using lemma (2.2) we get:

$$\widehat{f * h}(l,\alpha) = \int_{SO(n)} \left(\int_{S^{n-1}} h(R^{-1}\xi) \overline{(F_{l,n})(F_{j,n}) C_l^{\alpha} C_j^{\alpha}(\xi)} dS_{n-1}(\xi) \right) f(RN) dR$$

where N is the north pole. This implies that from the addition Lemma 2.1,

$$\widehat{f * h}(l,\alpha) = \int_{SO(n)} f(RN) \left(\int_{S^{n-1}} h(R^{-1}\xi) \sum_{|j| \le d_l(n)} d_l(n) C_j^{\alpha}(R\xi) F_{j,n}(R^{-1}\xi) dS_{n-1}(\xi) \right) C_l^{\alpha}(R^{-1}\xi) dR$$

which is 0 unless $\alpha = 0$. This follows since the measure on $SO(n)$ is rotation-invariant.
Therefore,

$$\widehat{f * h}(l,\alpha) = \int_{SO(n)} f(RN) C_l^{\alpha}(R^{-1}\xi) dR \hat{h}(l,0).$$

Finally using the relationship between C_l^{α} and the hyperspherical harmonics $P_{l,n}$ expressed in 15, we conclude that 24 holds.

In computing a fundamental solution, u_l say, of Laplace's equation on S^n we know that:

$$\Delta_{S^{n-1}} u_l(\xi,\xi') = \delta(\xi,\xi'), \qquad (25)$$

where $\delta(\xi,\xi')$ is the Dirac delta distribution on the manifold S^n. The Dirac delta distribution on the Riemannian manifold S^n is defined for an open set $U \subset S^n$ with $\xi, \xi' \in S^n$ such that:

$$\int_U \delta(\xi,\xi') dV_n = \begin{cases} 1 & \text{if } \xi' \in U, \\ 0 & \text{if } \xi' \notin U. \end{cases}$$

Using the standard hyperspherical coordinates (1) on S^n, the Dirac delta distribution is given by

$$\delta(\xi,\xi') = \frac{\delta(\theta_{n-1}-\theta'_{n-1})\delta(\theta_1-\theta_1')\delta(\theta_3-\theta_3')\cdots\delta(\theta_{n-3}-\theta_{n-3}')}{\sin^{n-1}\theta'_{n-2}\sin\theta'_{n-3}\cdots\sin^{n-3}\theta'_{n-2}}. \qquad (26)$$

This gives us another main result of this work as the next Theorem.

Theorem 3.5: The Gegenbauer harmonics C_l^{ν} are complete and there exists a fundamental solution ψ of the spherical Laplace-Beltrami Eq. 16 on S^n which admits Gegenbauer filtering.

Proof. The completeness relation for hyperspherical harmonics in standard hyperspherical coordinates 1 follows from their orthogonality:

$$\sum_{l=0}^{\infty} \sum_{\nu} C_l^{\nu}(\theta_1,\theta_2,\cdots,\theta_{n-1}) \overline{C_l^{\nu}(\theta'_1,\theta'_2,\cdots,\theta'_{n-1})}$$
$$= \frac{\delta(\theta_1-\theta'_1)\delta(\theta_2-\theta'_2)\cdots\delta(\theta_{n-1}-\theta'_{n-1})}{\sin^{n-2}\theta'_{n-1}\cdots\sin\theta'_2}.$$

Therefore, through 26, we can write,

$$\delta(\xi,\xi) = \frac{\delta(\theta-\theta\prime)}{\sin^{n-1}\theta\prime}\sum_{l=0}^{\infty}\sum_{\nu} C_l^{\gamma}(\theta_1,\theta_2,\cdots,\theta_{n-1})\overline{C_l^{\gamma}(\theta'_1,\theta'_2,\cdots,\theta'_{n-1})}. \quad (27)$$

Moreover since $\mathcal{G}_{l,n}$ is harmonic on its domain for fixed $\theta_1,\theta_2,\theta_3,\cdots,\theta_{n-2}\in[0,\pi]$, its restriction is in $C^2(S^{n-1})$ and therefore has a unique expansion in hyperspherical harmonics, namely

$$(\mathcal{G}_{l,n} * C_l^{\gamma})(\xi,\xi') = \sum_{l=0}^{\infty}\sum_{\nu} u_l^{\gamma}(\theta_1,\theta_2,\cdots,\theta_{n-1})C_l^{\gamma}(\theta_1,\theta'_1,\cdots,\theta'_{n-2}), \quad (28)$$

where $u_l^{\gamma}:[0,\pi]^n\to\mathbb{C}$. Furthermore, substitute 27, 28 into 25 and use the definition of Δ_{S^n} satisfying 17 to obtain:

$$\sum_{l=0}^{\infty}\sum_{\nu} C_l^{\gamma}(\theta_1,\theta_2,\cdots,\theta_{n-1})\left[\frac{\partial^2}{\partial\theta^2}+(n-1)\frac{\partial}{\partial\theta}\frac{l(l+n-2)}{\sin^2\theta}\right]\psi_l^{\gamma}(\theta,\theta',\theta'_2,\cdots,\theta'_{n-1})$$
$$=\sum_{l=0}^{\infty}\sum_{\nu}\psi_l^{\gamma}(\theta_1,\theta_2,\cdots,\theta_{n-1})C_l^{\gamma}(\theta_1,\theta'_1,\cdots,\theta'_{n-1})\frac{\delta(\theta-\theta\prime)}{\sin^{n-1}\theta\prime}.$$

This indicates the existence of $\psi_l:[0,\pi]^2\to\mathbb{R}$ such that

$$\psi_l^{\gamma}(\theta_1,\theta_2,\cdots,\theta_{n-1}) = \psi_l(\theta,\theta')\overline{C_l^{\gamma}(\theta'_1,\theta'_2,\cdots,\theta'_{n-1})}.$$

From 28, the expression for a fundamental solution of the Laplace-Beltrami operator in standard hyperspherical coordinates on the hypersphere is therefore given by:

$$\mathcal{G}_{l,n} * \psi(\xi,\xi') = \sum_{l=0}^{\infty}\sum_{\nu}\psi_l(\theta,\theta')\sum_{\nu} C_l^{\gamma}(\theta_1,\theta'_1,\cdots,\theta'_{n-1})\overline{C_l^{\gamma}(\theta'_1,\cdots,\theta'_{n-1})} = u_l. \quad (29)$$

Using the addition theorem for hyperspherical harmonics, Eq. 29 can now be simplified.

Therefore,

$$\mathcal{G}_{l,n} * \psi(\xi,\xi') = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-2)}\sum_{l=0}^{\infty}\psi_l(\theta_{n-1},\theta_{n-1}{}')(2l+n-2)C_l^{\frac{n}{2}-1}(\cos\gamma)$$

where γ is the geodesic angle between ξ and ξ'.

As an application, consider the Cauchy wave problem:

$$u_{tt} = u_{xx};\ \ x\in\mathbb{R}, \qquad t>0$$

$$u(x,0) = f(x) = \begin{cases} x & \text{if } 0\le x<1,\\ 2-x & \text{if } 1\le x\le 2,\\ 0 & \text{if otherwise;}\end{cases}$$

$$u_t(x,0) = g(x) = 0.$$

By the d'Alembert formula, (Pinchover and Rubinstein, 2005),

$$u(x,t) = \frac{1}{2}f(x+t)+\frac{1}{2}f(x-t).$$

With the aid of MATHEMATICA, we compute and plot the Gegenbauer filtration of u at $t=1$ and $t=2$ using the synthesized product of the real parts of $C_2^1(t)$ and $Y_{2,1}(t,\phi)$ shown in Fig. 1.

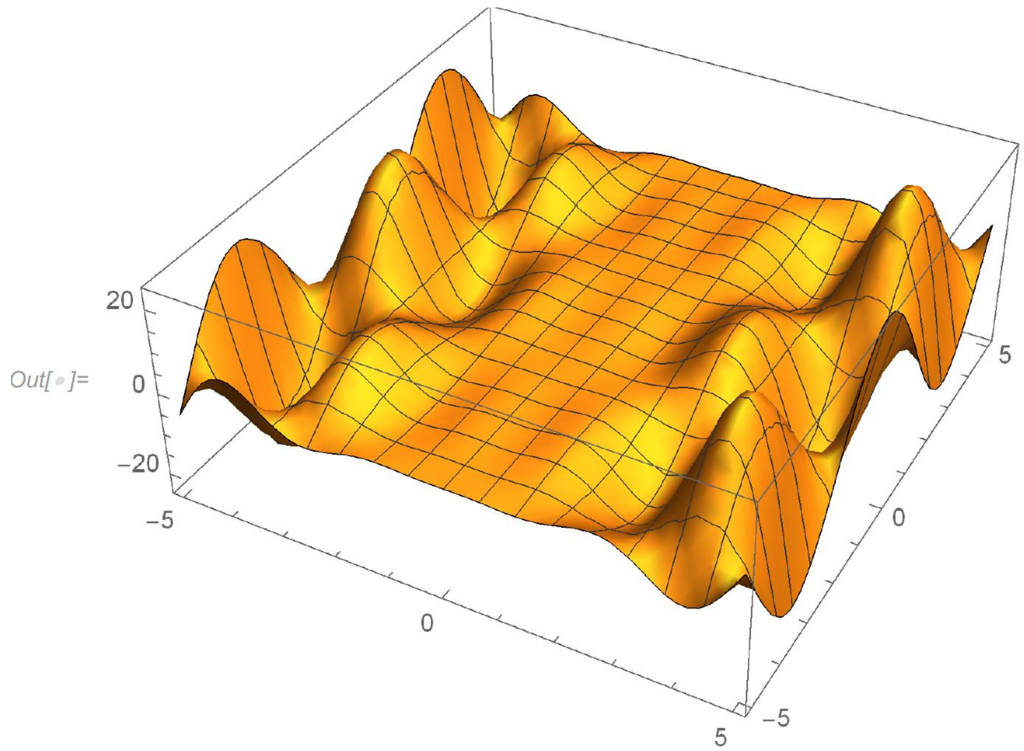


Fig. 1: The synthesized smoothing kernel $C_2^1(t) * Y_{2,1}(t,\phi)$

Fig. 1 is the synthesized smoothing kernel $C_2^1(t) * Y_{2,1}(t,\phi)$.

For $t = 1$,

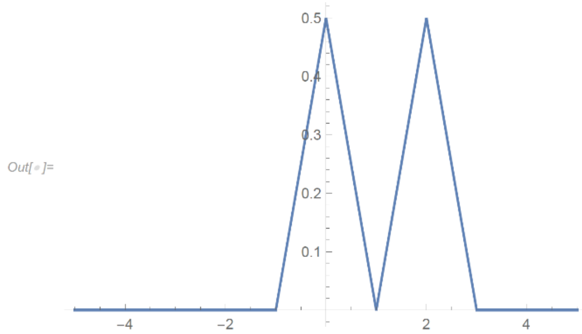
$$u(x,1) = \begin{cases} \frac{1}{2}(x+1) & \text{if } -1 \le x < 0, \\ \frac{1}{2}(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{if otherwise} \end{cases} + \begin{cases} \frac{1}{2}(x-1) & \text{if } 1 \le x < 2, \\ \frac{1}{2}(3-x) & \text{if } 2 \le x \le 3, \\ 0 & \text{if otherwise.} \end{cases}$$

and so we have the profile sketch as Figs. 2A, 2B, and 2C.

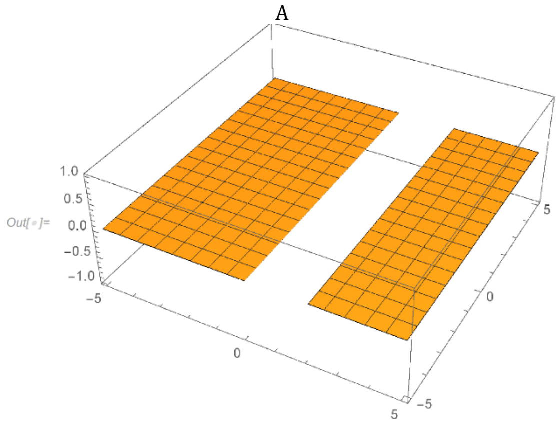
Fig. 2 are the wave profile at $t = 1$, the synthesised profile at $t = 1$ and Gegenbauer convolved output respectively. For $t = 2$,:

$$u(x,2) = \begin{cases} \frac{1}{2}(x+2) & \text{if } -2 \le x < -1, \\ \frac{1}{2}(-x) & \text{if } -1 \le x \le 1, \\ 0 & \text{if otherwise} \end{cases} + \begin{cases} \frac{1}{2}(x-1) & \text{if } 2 \le x < 3, \\ \frac{1}{2}(3-x) & \text{if } 3 \le x \le 4, \\ 0 & \text{if otherwise.} \end{cases}$$

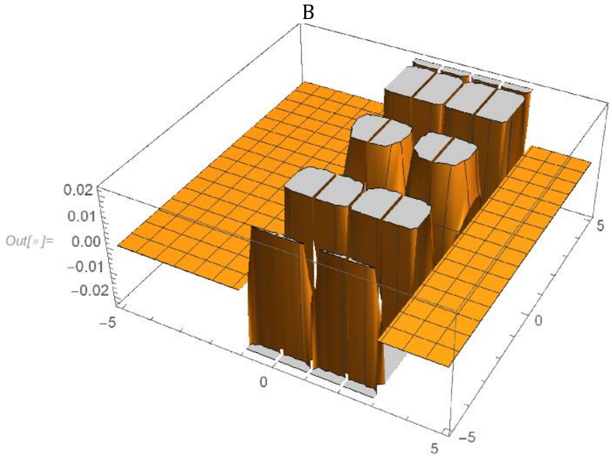
and so we have the profile sketch as Figs. 3A, 3B and 3C.



A



B



C

Fig. 2: Profile sketch

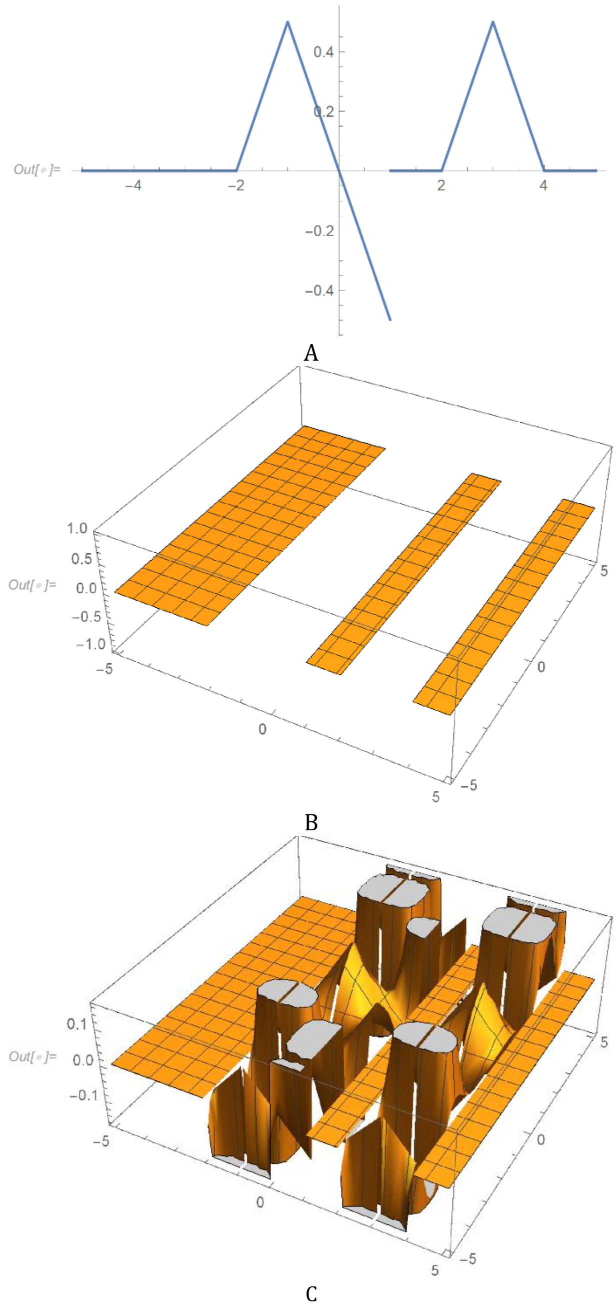


Fig. 3: Gegenbauer convolved output profile sketch

The captions Fig. 3 are the wave profile at t = 2, the synthesized profile at t = 2 and the Gegenbauer convolved output, respectively.

4. Conclusion

We studied the Gegenbauer kernel filtration of harmonic functions on the hypersphere. We showed that under the filtration, an explicit expression for the fundamental solution of a Laplace-type operator on a Riemannian manifold can be constructed. A demonstration of the Gegenbauer filtration kernel with a closed-form fundamental solution was shown. This brings to the limelight an extension of signal processing methods on Euclidean spaces to non-Euclidean spaces of higher dimensions such as closed Riemannian manifolds, for example, the hypersphere.

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Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

References

Antoine JP and Vandergheynst P (1999). Wavelets on the 2-sphere: A group-theoretical approach. Applied and Computational Harmonic Analysis, 7(3): 262-291. https://doi.org/10.1006/acha.1999.0272

Assche WV, Yánez RJ, González-Férez R, and Dehesa JS (2000). Functionals of Gegenbauer polynomials and d-dimensional hydrogenic momentum expectation values. Journal of Mathematical Physics, 41(9): 6600-6613. https://doi.org/10.1063/1.1286984

Atkinson K and Han W (2012). Spherical harmonics and approximations on the unit sphere: An introduction. Volume 2044, Springer Science and Business Media, Berlin, Germany. https://doi.org/10.1007/978-3-642-25983-8

Aubin T (1998). Some nonlinear problems in Riemannian geometry. Springer, Berlin, Germany. https://doi.org/10.1007/978-3-662-13006-3

Avery JS (2012). Hyperspherical harmonics: Applications in quantum theory. Volume 5, Springer Science and Business Media, Berlin, Germany.

Bezubik A and Strasburger A (2006). A new form of the spherical expansion of zonal functions and Fourier transforms of SO (d)-finite functions. Symmetry, Integrability and Geometry: Methods and Applications, 2: 033. https://doi.org/10.3842/SIGMA.2006.033

Bogdanova I, Vandergheynst P, Antoine JP, Jacques L, and Morvidone M (2005). Stereographic wavelet frames on the sphere. Applied and Computational Harmonic Analysis, 19(2): 223-252. https://doi.org/10.1016/j.acha.2005.05.001

Bulow T (2004). Spherical diffusion for 3D surface smoothing. IEEE Transactions on Pattern Analysis and Machine Intelligence, 26(12): 1650-1654. https://doi.org/10.1109/TPAMI.2004.129 **PMid:15573826**

Camporesi R (1990). Harmonic analysis and propagators on homogeneous spaces. Physics Reports, 196(1-2): 1-134. https://doi.org/10.1016/0370-1573(90)90120-Q

Claessens SJ (2016). Spherical harmonic analysis of a harmonic function given on a spheroid. Geophysical Journal International, 206(1): 142-151. https://doi.org/10.1093/gji/ggw126

Cohl HS and Palmer RM (2015). Fourier and Gegenbauer expansions for a fundamental solution of Laplace's equation in hyperspherical geometry. Symmetry, Integrability and Geometry: Methods and Applications, 11: 015. https://doi.org/10.3842/SIGMA.2015.015

Dai F and Xu Y (2013). Convolution operator and spherical harmonic expansion. In: Dai F and Xu Y (Eds.), Approximation theory and harmonic analysis on spheres and balls: 29-51. Springer, New York, USA. https://doi.org/10.1007/978-1-4614-6660-4_2 **PMCid:PMC3569964**

Drake JB, Worley P, and D'Azevedo E (2008). Spherical harmonic transform algorithms. ACM Transactions on Mathematical Software, 35(3): 111-131. https://doi.org/10.1145/1391989.1404581

Driscoll JR and Healy DM (1994). Computing Fourier transforms and convolutions on the 2-sphere. Advances in Applied Mathematics, 15(2): 202-250. https://doi.org/10.1006/aama.1994.1008

Healy DM, Rockmore DN, Kostelec PJ, and Moore S (2003). FFTs for the 2-sphere-improvements and variations. Journal of Fourier Analysis and Applications, 9(4): 341-385. https://doi.org/10.1007/s00041-003-0018-9

Jost J and Jost J (2008). Riemannian geometry and geometric analysis. Volume 42005, Springer, Berlin, Germany.

Lee JM (2003). Introduction to smooth manifolds: Graduate texts in mathematics. Springer Science and Business Media, New York, USA. https://doi.org/10.1007/978-0-387-21752-9

Morimoto M (1998). Analytic functionals on the sphere. Translations of Mathematical Monographs, American Mathematical Society, Providence, USA. https://doi.org/10.1090/mmono/178

Omenyi L (2014). On the second variation of the spectral zeta function of the Laplacian on homogeneous Riemanniann manifolds. Ph.D. Dissertation, Loughborough University, Loughborough, UK.

Omenyi L and Uchenna M (2019). Global analysis on Riemannian manifold. The Australian Journal of Mathematical Analysis and Applications, 16(2): 1-17.

Pinchover Y and Rubinstein J (2005). An introduction to partial differential equations. Volume 10, Cambridge University Press, Cambridge, USA. https://doi.org/10.1017/CBO9780511801228

Strasburger A (1993). A generalization of the Bochner identity. Expositiones Mathematicae, 11: 153-157.

Szekeres P (2004). A course in modern mathematical physics: Groups, Hilbert space and differential geometry. Cambridge University Press, Cambridge, UK. https://doi.org/10.1017/CBO9780511607066

Wong MW (2006). Weyl transforms, heat kernels, green functions and Riemann zeta functions on compact lie groups. In: Toft J, Wong MW, and Zhu H (Eds.), Modern trends in Pseudo-differential operators: 67-85. Birkhäuser, Basel, Switzerland. https://doi.org/10.1007/978-3-7643-8116-5_4