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The square root of tridiagonal Toeplitz matrices

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ABSTRACT

In this paper, we present an explicit formula to find square roots of a tridiagonal Toeplitz matrix, and we show that these square roots have the form of a persymmetric matrix with examples to illustrate.

1. Introduction

The tridiagonal Toeplitz matrix A is illustrated below:

$$A = \begin{pmatrix} b & a & & & \\ c & b & a & & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \\ & & & c & b \end{pmatrix}_{n\times n} \quad with \quad a \neq 0 \ and \ c \neq 0 \tag{1.1}$$

This type of matrix is used in several different fields of applications, such as a solution of ordinary and partial differential equations, time series analysis, and regularization matrices in Tikhonov regularization for the solution of discrete ill-posed problems. It is, therefore, important to understand the properties of this type of matrix (Noschese et al., 2013).

Yuttanan and Nilrat (2005) gave an answer to the question of which matrices have an n^{th} root for any positive integer n and which have an n^{th} root only for some positive integer n. As a special case the diagonalizable matrices always have the n^{th} roots.

In Section 2 and through the work of Salkuyeh (2006) about positive integer powers of the tridiagonal Toeplitz matrices, we give a method to calculate square roots of this type of matrix with proof the form of the square root in section 3 with examples to illustrate.

2. The square root of tridiagonal Toeplitz matrices

In this section, we will give an explicit statement to the square root of the tridiagonal Toeplitz matrix, and we start with present some important results.

Theorem 1: Let B be a complex matrix of order m. If B is diagonalizable, then B has an n^{th} root for any positive integer n.

Proof: see Yuttanan and Nilrat (2005).

As a special case, the matrix B has a square root, i.e., there exists a matrix R such that: $R^2 = B$ (Yuttanan and Nilrat, 2005).

Lemma 1: Let A be a tridiagonal Toeplitz matrix defined in (1.1), the eigenvalues and eigenvectors of A are given by,

$$\lambda_j = b + 2a\sqrt{\frac{c}{a}}\cos\left(\frac{j\pi}{n+1}\right) \tag{2.1}$$

and,

$$x_j = \begin{pmatrix} \left(\frac{c}{a}\right)^{\frac{1}{2}} sin\left(\frac{1j\pi}{n+1}\right) \\ \left(\frac{c}{a}\right)^{\frac{2}{2}} sin\left(\frac{2j\pi}{n+1}\right) \\ \left(\frac{c}{a}\right)^{\frac{3}{2}} sin\left(\frac{3j\pi}{n+1}\right) \\ \vdots \\ \left(\frac{c}{a}\right)^{\frac{n}{2}} sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix} \tag{2.2}$$

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Moreover, the matrix A is diagonalizable, i.e., there exists an invertible matrix P such that:

$$A = PDP^{-1} \tag{2.3}$$

with,

$$P = (x_1 \quad x_2 \quad \cdots \quad x_n) \quad and \quad D = diag(\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n) \tag{2.4}$$

Proof: See Meyer (2000).

Using (2.1), (2.2), (2.3), and theorem (1), we can give an explicit expression for a square root of the matrix defined in (1.1) (Meyer, 2000).

From the work of Salkuyeh (2006), we have:

$$P^{-1} = \tilde{P}^{-1}\tilde{D}^{-1} \tag{2.5}$$

$$\tilde{P}^{-1} = \frac{2}{n+1}\tilde{P} \tag{2.6}$$

with

$$\tilde{D} = diag\left(\left(\frac{c}{a}\right)^{\frac{1}{2}} \quad \left(\frac{c}{a}\right)^{\frac{2}{2}} \quad \cdots \quad \left(\frac{c}{a}\right)^{\frac{n}{2}}\right), \tilde{P} = (\tilde{x}_1 \quad \tilde{x}_2 \quad \cdots \quad \tilde{x}_n)$$

in which

$$\tilde{D} = diag\left(\left(\frac{c}{a}\right)^{\frac{1}{2}}, \left(\frac{c}{a}\right)^{\frac{2}{2}}, \cdots \left(\frac{c}{a}\right)^{\frac{n}{2}}\right)$$

$$\tilde{x}_j = \begin{pmatrix} sin\left(\frac{1j\pi}{n+1}\right) \\ sin\left(\frac{2j\pi}{n+1}\right) \\ sin\left(\frac{3j\pi}{n+1}\right) \\ \vdots \\ sin\left(\frac{nj\pi}{n+1}\right) \end{pmatrix} \tag{2.7}$$

For more details, See Salkuyeh (2006).

Theorem 2: Let A be a tridiagonal Toeplitz matrix defined in (1.1), then a square root $R = (r_{i\,j})$ of A is given by,

$$r_{i\,j} = \frac{2}{n+1}\left(\frac{c}{a}\right)^{\frac{i-j}{2}} \sum_{k=1}^{n} \sqrt{\lambda_k} \sin\frac{ik\pi}{n+1} \sin\frac{jk\pi}{n+1} \tag{2.8}$$

where

$$\lambda_j = b + 2a\sqrt{\frac{c}{a}}\cos\left(\frac{j\pi}{n+1}\right).$$

Proof: From (2.3), the matrix A defined in (1.1) is diagonalizable. From theorem (1), there exists a matrix R such that: $R^2 = A$.

Let:

$$\sqrt{D} = diag(\sqrt{\lambda_1} \quad \sqrt{\lambda_2} \quad \cdots \quad \sqrt{\lambda_n}). \tag{2.9}$$

Therefore, we have:

$$R = P\sqrt{D}P^{-1} \tag{2.10}$$

where P and P^{-1} were defined in (2.4) and (2.5). Hence,

$$R^2 = P\sqrt{D}P^{-1}P\sqrt{D}P^{-1} = PDP^{-1} = A$$

and,

$$\begin{aligned} R &= P\sqrt{D}P^{-1} \\ &= P\sqrt{D}\tilde{P}^{-1}\tilde{D}^{-1} && using\ (2.5) \\ &= \frac{2}{n+1}P\sqrt{D}\tilde{P}\tilde{D}^{-1} && using\ (2.6) \\ &= \frac{2}{n+1}P\ diag(\sqrt{\lambda_1} \quad \sqrt{\lambda_2} \quad \cdots \quad \sqrt{\lambda_n})\ \tilde{P}\ diag\left(\left(\frac{c}{a}\right)^{-\frac{1}{2}} \quad \left(\frac{c}{a}\right)^{-\frac{2}{2}} \quad \cdots \quad \left(\frac{c}{a}\right)^{-\frac{n}{2}}\right) \end{aligned}$$

using (2.2) and (2.7), the relation (2.8) is obtained.

Example 1: Let:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

be a tridiagonal Toeplitz matrix with $a = 1$, $b = 2$, $c = 2$. By using lemma (1) and (2.1), we have $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$. Then from theorem (2), we have:

$$r_{i\,j} = (2)^{\frac{i-j-2}{2}} \sum_{k=1}^{n} \sqrt{\lambda_k} \sin\frac{ik\pi}{n+1} \sin\frac{jk\pi}{n+1}, \qquad i,j = 1,2,3$$

then we have:

$$R = \begin{pmatrix} \frac{1+\sqrt{2}}{2} & \frac{1}{2} & \frac{1-\sqrt{2}}{4} \\ 1 & 1 & \frac{1}{2} \\ 1-\sqrt{2} & 1 & \frac{1+\sqrt{2}}{2} \end{pmatrix}$$

3. Structure of square root of tridiagonal Toeplitz matrices

In this section, we prove that the square root of the tridiagonal Toeplitz matrix defined in (2.8) takes the form of a persymmetric matrix.

Let's start with the definition of a persymmetric matrix.

Definition 1: Let $A = (a_{i\,j})$ be an $n \times n$ matrix. A is said to be a persymmetric matrix if it is symmetric across its lower-left to upper-right diagonal:

$$a_{i\,j} = a_{n-j+1\,n-i+1} \quad for \quad i,j = 1,\cdots,n.$$

For example, 6-by-6 persymmetric matrices are of the form:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{15} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{24} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{33} & a_{23} & a_{13} \\ a_{51} & a_{52} & a_{42} & a_{32} & a_{22} & a_{12} \\ a_{61} & a_{51} & a_{41} & a_{31} & a_{21} & a_{11} \end{pmatrix}$$

Theorem 3: The square root $R = (r_{i\,j})$ of tridiagonal Toeplitz matrices defined in (2.8) is a persymmetric matrix:

$$r_{i\,j} = r_{n-j+1\,n-i+1} \quad for \ i,j \in \{1,2,\cdots,n\} \tag{3.1}$$

Proof: To prove (3.1), you must note the following:

$$\sin(k\pi - \theta) = \begin{cases} \sin(\theta) & if \quad k = 2p+1 \\ -\sin(\theta) & if \quad k = 2p \end{cases} = (-1)^{k+1}\sin(\theta) \qquad \forall k \in \mathbb{N} \tag{3.2}$$

Now by using (2.8), we have:

$$\begin{aligned} r_{n-j+1\,n-i+1} &= \frac{2}{n+1}\left(\frac{c}{a}\right)^{\frac{n-j+1-n+i-1}{2}} [k = 1]n \sum \sqrt{\lambda_k}\sin\left(\frac{(n-j+1)k\pi}{n+1}\right)\sin\left(\frac{(n-i+1)k\pi}{n+1}\right) \\ &= \frac{2}{n+1}\left(\frac{c}{a}\right)^{\frac{i-j}{2}} \sum_{k=1}^{n} \sqrt{\lambda_k}\sin\left(k\pi - \frac{jk\pi}{n+1}\right)\sin\left(k\pi - \frac{ik\pi}{n+1}\right) \\ &= \frac{2}{n+1}\left(\frac{c}{a}\right)^{\frac{i-j}{2}} \sum_{k=1}^{n} \sqrt{\lambda_k}\,(-1)^{k+1}\sin\left(\frac{jk\pi}{n+1}\right)(-1)^{k+1}\sin\left(\frac{ik\pi}{n+1}\right) \qquad using \ (3.2) \\ &= r_{i\,j} \end{aligned}$$

Example 2: Let:

$$B = \begin{pmatrix} 3 & 1 & & \\ 2 & 3 & 1 & \\ & 2 & 3 & 1 \\ & & 2 & 3 \end{pmatrix}$$

By using (2.8), we have:

$$R \approx \begin{pmatrix} 1.6740 & 0.3109 & -0.0316 & 0.0059 \\ 0.6218 & 1.6108 & 0.3226 & -0.0316 \\ -0.1266 & 0.6453 & 1.6108 & 0.3109 \\ 0.0470 & -0.1266 & 0.6218 & 1.6740 \end{pmatrix}$$

Observe that R is a persymmetric matrix.

4. Conclusion

Based on this article, we obtain a formula for calculating a square root of a tridiagonal Toeplitz matrix, which is diagonalizable. The reader can find other properties and other square roots of other diagonalizable matrices with the same method we used.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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