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A study on ordered AG-groupoids by their fuzzy interior ideals

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ABSTRACT

The purpose of this paper is to study the different classes of ordered AG-groupoids by using fuzzy left (resp. right, interior) ideals. Particularly, we illustrate regular (resp. right regular, left regular, (2, 2)-regular, weakly regular and intra-regular) ordered AG-groupoids in terms of fuzzy left (resp. right, interior) ideals. In this regard, we show that in (regular, right regular, weakly regular) ordered AG-groupoids, the concept of fuzzy (interior, two-sided) ideals coincide. The concept of fuzzy (interior, two-sided) ideals coincide in ((2,2), left, intra-) regular ordered AG-groupoids with left identity.

1. Introduction

In 1972, a generalization of commutative semigroup had been established by Kazim and Naseeruddin (1977). In ternary commutative law: $abc = cba$, they introduced braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This they called the left invertive law. A groupoid S is said to be left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$. Holgate (1992) has called the same structure as left invertive groupoid. This structure is also known as Abel-Grassmann's groupoid (abbreviated as AG-groupoid). In fact, an AG-groupoid is a non-commutative and non-associative semigroup. It is a midway structure between a commutative semigroup and a groupoid. Ideals in AG-groupoids have been investigated by Mushtaq and Yusuf (1978). A groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $((ab)(cd) = (db)(ca))$. An AG-groupoid is medial, but in general, an AG-groupoid needs not to be paramedial. Every AG-groupoid with left identity is paramedial and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Algebraic structures play a prominent role in mathematics with wide-ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, and the like. Although semigroups concentrate on theoretical aspects, they also include applications in error-correcting codes, control engineering, formal language, computer science, and information science.

Algebraic structures, especially ordered semigroups play a prominent role in mathematics with wide-ranging applications in many disciplines such as control engineering, computer arithmetics, coding theory, sequential machines, and formal languages.

$(S,\cdot,\leq)$ is an ordered semigroup and $A \subseteq S$, we denote by $(A]$, the subset of S defined as follows: $(A] = \{s \in S \ : \ s \leq a \text{ for some } a \in A\}$. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$.

The notions of ideals play a crucial role in the study of (ring, semi-ring, near-ring, semigroup, ordered semigroup) theory, etc.

A non-empty subset A of S is called a left (resp. right) ideal of S if the following hold (1) $SA \subseteq A$ (resp. $AS \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$. Equivalent definition: A is called a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$). A non-empty subset A of S is called an interior ideal of S if (1) $SAS \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$.

An ordered semigroup S is said to be regular if, for every $a \in S$, there exists $x \in S$ such that $a \leq axa$. Equivalent definitions are as follows: (1) $A \subseteq (ASA]$ for every $A \subseteq S$. (2) $a \in (aSa]$ for every $a \in S$.

An ordered semigroup S is said to be (2,2)-regular, if for every $a \in S$, there exists $x \in S$ such that $a \leq a^2xa^2$Equivalent definitions are as follows:

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(1) $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. (2) $a \in (a^2Sa^2]$ for every $a \in S$.

An ordered semigroup S is said to be weakly regular, if for every $a \in S$, there exist $x, y \in S$ such that $a \leq axay$. Equivalent definitions are as follows: (1) $A \subseteq ((AS)^2]$ for every $A \subseteq S$. (2) $a \in ((aS)^2]$ for every $a \in S$.

An ordered semigroup S is an intra-regular, if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. Equivalent definitions are as follows: (1) $A \subseteq (SA^2S]$ for every $A \subseteq S$. (2) $a \in (Sa^2S]$ for every $a \in S$.

We define fuzzy left (resp. right, interior) ideals in ordered AG-groupoids, basically, an ordered AG-groupoid is non-commutative and non-associative ordered semigroup.

In this present paper, we characterize regular (resp. right regular, left regular, (2, 2) -regular, weakly regular, and intra-regular) ordered AG-groupoids in terms of fuzzy left (resp. right, interior) ideals. In this regard, we prove that in regular, right regular, weakly regular) ordered AG-groupoids, the concept of fuzzy (interior, two-sided) ideals coincide. The concept of fuzzy (interior, two-sided) ideals coincide in ((2, 2) left, intra-) regular ordered AG-groupoids with left identity.

2. Preliminaries

By Shah and Kausar (2014), an ordered AG-groupoid S, is a partially ordered set, at the same time, an AG-groupoid such that $a \leq b$, implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. Two conditions are equivalent to the one condition $(ca)d \leq (cb)d$ for all $a, b, c, d \in S$. An ordered AG-groupoid is also called a po-AG-groupoid for short.

Let S be an ordered AG-groupoid and $A \subseteq S$, we define a subset $(A] = \{s \in S : s \leq a \text{ for some } a \in A\}$ of S and obviously $A \subseteq (A]$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $A, B \subseteq S$, then $AB = \{ab \mid a \in A, b \in B\}$, $((A]] = (A]$, $(A](B] \subseteq (AB]$, $((A](B]] = (AB]$, if $A \subseteq B$ then $(A] \subseteq (B]$, $(A \cap B] \neq (A] \cap (B]$ in general.

For $\emptyset \neq A \subseteq S$. A is called an AG-subgroupoid of S if $A^2 \subseteq A$. A is called a left left (resp. right) ideal of S if the following hold (1) $SA \subseteq A$(resp. $AS \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$.

Equivalent definition: A is called a left left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ $(resp. AS \subseteq A)$. A is called an ideal of S if A is both a left and a right ideal of S. If A, B are ideals of S then $A \cup B$ and $A \cap B$ are also ideals of S.

A non-empty subset A of S is called an interior ideal of S if (1) $(SA)S \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leq a$ implies $b \in A$ $(or\ (A] \subseteq A)$.

An ordered AG-groupoid S is said to be regular if, for every $a \in S$, there exists $x \in S$ such that $a \leq (ax)a$. Equivalent definitions are as follows: (1) $A \subseteq ((AS)A]$ for every $A \subseteq S$. (2) $a \in ((aS)a]$ for every $a \in S$.

An ordered AG-groupoid S is left (resp. right) regular if for every $a \in S$, there exists $x \in S$ such that $\leq xa^2$ $(resp. a \leq a^2x)$. Equivalent definitions are as follows: (1) $A \subseteq (SA^2]$ (resp. $A \subseteq (A^2S]$) for every $A \subseteq S$. (2) $a \in (Sa^2]$ (resp. $a \in (a^2S]$) for every $a \in S$.

An ordered AG-groupoid S is said to be completely regular if it is regular, left regular, right regular.

An ordered AG-groupoid S is said to be strongly regular if, for every $a \in S$, there exists $x \in S$ such that $a \leq (ax)a$ and $ax = xa$.

Every strongly regular ordered AG-groupoid is right regularly ordered AG-groupoid.

An ordered AG-groupoid S is said to be weakly regular if for every $a \in S$, there exist $x, y \in S$ such that $a \leq (ax)(ay)$. Equivalent definitions are as follows: (1)$A \subseteq ((AS)^2]$ for every $A \subseteq S$. (2) $a \in ((aS)^2]$ for every $a \in S$.

An ordered AG-groupoid S is called intra-regular if for every $a \in S$, there exist $x, y \in S$ such that $a \leq (xa^2)y$. Equivalent definitions are as follows: (1) $A \subseteq ((SA^2)S]$ for every $A \subseteq S$. (2) $a \in ((Sa^2)S]$ for every $a \in S$.

We denote by $L(a), R(a), I(a)$ the left ideal, the right ideal, and the ideal of S, respectively generated by a. We have $L(a) = \{s \in S : s \leq a \text{ or } s \leq xa \text{ for some } \in S\} = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup (Sa)S]$.

Example 1: Let $S = \{a, b, c, d, e\}$. Define multiplication $\cdot$ in S as follows:

.	a	B	c	d	e
a	a	A	a	a	a
b	a	A	a	a	a
c	a	A	e	c	d
d	a	A	d	e	c
e	a	A	c	d	E

and $\leq$⥂$:= \{(a, a), (a, b), (b, a), (e, e)\}$. Then S is an ordered AG-groupoid and$A = \{c, d, e\}$ is an AG-subgroupoid of S and $I = \{a, c, d, e\}$ is an ideal of S.

Remark 1: Every ideal (whether right, left, or two-sided) is an AG-subgroupoid, but the converse is not true in general. An ordered AG-groupoid S is said to be locally associative if for every $\in S$, $(a.a).a = a.(a.a)$.

Example 2: Let $S = \{a, b, c\}$. Define multiplication $\cdot$ in S as follows:

.	A	b	c
a	C	c	b
b	B	b	b
c	B	b	b

and $\leq := \{(a, a), (b, b), (c, c)\}$. Then $(S, \cdot, \leq)$ is a locally associative ordered AG-groupoid.

In a locally associative ordered AG-groupoids S, we define powers of an element as follow: ⥂ $a^1 = a$, $a^{n+1} = a^n a$. If S has a left identity e, we define $a^0 = e$, as left identity is unique in an ordered AG-groupoid. A locally associative ordered AG-groupoid S with left identity e has associative powers.

3. Fuzzy interior ideals on ordered AG-groupoids

A fuzzy set μ of a given set X is described as an arbitrary function $\mu : X \to [0,1]$, where $[0,1]$ is the unit closed interval of real numbers.

The fundamental concept of a fuzzy set, introduced by Zadeh (1965), which gives a natural framework for the generalizations of some basic notions of algebra, for example, set (resp. group, semigroup, ring, near-ring, semi-ring) theory, groupoids, real analysis, topology, differential equations and so forth. Rosenfeld (1971) introduced the concept of a fuzzy set in the group. The study of the fuzzy set in semigroup investigated by Kuroki (1995). He studied fuzzy ideals and fuzzy (interior, bi-, quasi-, quasi-semiprime) ideals in semigroups. Dib and Galhum (1997) examined the definition of fuzzy groupoid (resp. semigroup). They studied fuzzy ideals and fuzzy bi-ideals of fuzzy semigroups.

A systematic exposition of fuzzy semigroups by Mordeson et al. (2003), where one can find theoretical results on fuzzy semigroups and their use in fuzzy finite state machines and fuzzy languages. Fuzzy sets in ordered semigroups/ordered groupoids established by Kehayopulu and Tsingelis (2002).

By a fuzzy set μ of an ordered AG-groupoid S, we mean a function $\mu : S \to [0,1]$ and the complement of μ is denoted by μ', is a fuzzy set in S given by $\mu'(x) = 1 - \mu(x)$ for all $x \in S$.

A fuzzy set μ of S is called a fuzzy AG-subgroupoid of S if $\mu(xy) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in S$. μ is called a fuzzy left (resp. right) ideal of S if (1) $\mu(xy) \geq \mu(y)$ (resp. $\mu(xy) \geq \mu(x)$). (2) $x \leq y$, implies $\mu(x) \geq \mu(y)$ for all $x, y \in S$. μ is a fuzzy ideal of S if μ is both a fuzzy left and a fuzzy right ideal of S.

Equivalently, μ is called a fuzzy ideal of S if (1) $\mu(xy) \leq max\{\mu(x), \mu(y)\}$. (2) $x \leq y$, implies $\mu(x) \geq \mu(y)$ for all $x, y \in S$.

Every fuzzy ideal (whether right, left, two-sided) is a fuzzy AG-subgroupoid, but the converse is not true in general.

A fuzzy set μ of S is called a fuzzy interior ideal of S if (1) $\mu((xa)y) \geq \mu(a)$. (2) $x \leq y$, implies $\mu(x) \geq \mu(y)$ for all $x, a, y \in S$.

We denote by $F(S)$, the set of all fuzzy subsets of S. We define an order relation " $\subseteq$ " on $F(S)$ such that $f \subseteq g$ if and only if $f(x) \leq g(x)$ for all $x \in S$. Then $(F(S), \circ, \subseteq)$ is an ordered AG-groupoid.

By the symbols $f \wedge g$ and $f \vee g$, we will mean the following fuzzy subsets:

$$(\forall x \in S)(f \wedge g \ : \ S \to [0,1], x \mapsto (f \wedge g)(x) = min\{f(x), g(x)\});$$
$$(\forall x \in S)(f \vee g \ : \ S \to [0,1], x \mapsto (f \vee g)(x) = max\{f(x), g(x)\}).$$

Let $a \in S$ and $\emptyset \neq A \subseteq S$, and we define a set $A_a = \{(y,z) \in S \times S \mid a \leq yz\}$. Let f and g be fuzzy subsets of S, the product $f \circ g$ of f and g is defined by:

$$f \circ g \ : \ S \to [0,1], a \mapsto f \circ g(a) = \begin{cases} \bigvee_{(y,z) \in A_a} min\{f(y), g(z)\} & if\ A_a \neq \varphi \\ 0 & if\ A_a = \varphi \end{cases}$$

For a non-empty family of fuzzy subsets $\{f_i\}_{i \in I}$, of S, the fuzzy subsets $\bigvee_{i \in I} f_i$ and $\bigwedge_{i \in I} f_i$ of S are defined as follows:

$$\bigvee_{i \in I} f_i : S \to [0,1], a \mapsto (\bigvee_{i \in I} f_i)(a) := \sup_{i \in I}\{f_i(a)\}$$
$$and\ \bigwedge_{i \in I} f_i : S \to [0,1], a \mapsto (\bigwedge_{i \in I} f_i)(a) := \inf_{i \in I}\{f_i(a)\}.$$

If I is a finite set, say $I = \{1, 2, \ldots n\}$, then clearly,

$$\bigvee_{i \in I} f_i(a) = max\{f_1(a), f_2(a), \ldots, f_n(a)\}$$
$$and\ \bigwedge_{i \in I} f_i(a) = min\{f_1(a), f_2(a), \ldots, f_n(a)\}.$$

For S, the fuzzy subsets 0 and 1 are defined as follows:

$$0 : S \to [0,1], x \mapsto 0(x) := 0.$$
$$1 : S \to [0,1], x \mapsto 1(x) := 1.$$

Clearly, the fuzzy subset 0 (resp. 1) of S is the least (resp. the greatest) element of the ordered set $(F(S), \leq)$. The fuzzy subset 0 is the zero element of $(F(S), \circ, \leq)$ (that is, $f \circ 0 = 0 \circ f = 0$ and $0 \leq f$ for every $f \in F(S)$).

For $\emptyset \neq A \subseteq S$, the characteristic function of A is denoted by χ_A and defined by,

$$\chi_A(a) = \begin{cases} 1 \ if\ a \in A \\ 0 \ if\ a \notin A \end{cases}.$$

An ordered AG-groupoid S can be considered a fuzzy subset of itself, and we write $S = \chi_S$, i.e., $S(x) = \chi_S(x) = 1$ for all $x \in S$. This implies that $S(x) = 1$ for all $x \in S$.

For $, B \subseteq S$, then $A \subseteq B$ if and only if $f_A \leq f_B$ and $f_A \circ f_B = f_{(AB)}$.

Let μ be a fuzzy subset of S, and then for all $t \in (0,1]$, we define a set $U(\mu; t) = \{x \in S \mid \mu(x) \geq t\}$, which is called an upper t level cut of μ and can be used to the characterization of μ.

Example 3: Let $S = \{a, b, c, d\}$. Define multiplication in S as follows:

.	A	b	c	d
a	A	a	a	a
b	A	a	a	a
c	A	a	d	a
d	A	a	c	d

and $\leq := \{(a,a), (a,b), (b,a), (d,d)\}$. Then S is an ordered AG-groupoid. Let μ be a fuzzy set of S such that $\mu(a) = \mu(c) = \mu(d) = 0.7$, $\mu(b) = 0$. Then μ is a fuzzy right ideal of S.

Lemma 1: Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then the characteristic function $\chi_{(A]}$ of $(A]$ is a fuzzy subset of S satisfying the condition $x \leq y \Rightarrow \chi_{(A]}(x) \geq \chi_{(A]}(y)$ for all $x, y \in S$.

Proof: By the definition, $\chi_{(A]}$ is a mapping of S into $\{0,1\}(\subseteq [0,1])$. Let $x \leq y$, $x, y \in S$. If $y \notin (A]$, by definition $\chi_{(A]}(y) = 0$, thus $\chi_{(A]}(x) \geq \chi_{(A]}(y)$. If $y \in (A]$, this implies that $\chi_{(A]}(y) = 1$. Since $y \in (A]$, so there exists $z \in A$ such that $y \leq z$. Thus $x \leq z$, i.e., $x \in (A]$ and $\chi_{(A]}(x) = 1$. Thus $\chi_{(A]}(x) \geq \chi_{(A]}(y)$.

Proposition 1: Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then $A = (A]$ if and only if a fuzzy subset χ_A of S has the property $x \leq y \Rightarrow \chi_A(x) \geq \chi_A(y)$ for all $x, y \in S$.

Proof: Suppose $A = (A]$, then the characteristic function χ_A of A is a fuzzy subset of S satisfying the condition $x \leq y \Rightarrow \chi_A(x) \geq \chi_A(y)$, by the Lemma MB. Conversely, let $x \in (A]$, this implies that there exists $y \in A$ such that $x \leq y$. By the given condition, we have $\chi_A(x) \geq \chi_A(y)$. Since $y \in A$, we have $\chi_A(y) = 1$. Thus $\chi_A(x) = 1$, i.e., $x \in A$. Hence $A = (A]$.

Lemma 2: Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then A is an AG-subgroupoid of S if and only if the characteristic function χ_A of A is a fuzzy AG-subgroupoid of S.

Proof: Suppose A is an AG-subgroupoid of S and $x, y \in S$. If $x, y \notin A$, by definition $\chi_A(x) = 0 = \chi_A(y)$. Thus $\chi_A(xy) \geq \chi_A(x) \wedge \chi_A(y)$. If $x, y \in A$, by definition $\chi_A(x) = 1 = \chi_A(y)$. $xy \in A$, A being an AG-subgroupoid of S, this implies that $\chi_A(xy) = 1$. Thus $\chi_A(xy) \geq \chi_A(x) \wedge \chi_A(y)$. Hence the characteristic function χ_A of A is a fuzzy AG-subgroupoid of S.

Conversely, let $y \in A^2$, $x, y \in A$. By definition of a characteristic function $\chi_A(x) = 1 = \chi_A(y)$. $\chi_A(xy) \geq \chi_A(x) \wedge \chi_A(y) = 1$, χ_A is a fuzzy AG-subgroupoid of S. This implies that $\chi_A(xy) = 1$, i.e., $xy \in A$. Hence A is an AG-subgroupoid of S.

Lemma 3: Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then A is a left (resp. right) ideal of S if and only if the characteristic function χ_A of A is a fuzzy left (resp. right) ideal of S.

Proof: Suppose A is a left ideal of S and $x, y \in S$ such that $x \leq y$, this implies that $A = (A]$, A being a left ideal of S. Then $\chi_A(x) \geq \chi_A(y)$, by the Proposition **1**. If $y \notin A$, by definition $\chi_A(y) = 0$. Thus $\chi_A(xy) \geq \chi_A(y)$. If $y \in A$, by definition $\chi_A(y) = 1$. $xy \in A$, A being a left ideal, so $\chi_A(xy) = 1$. Thus $\chi_A(xy) \geq \chi_A(y)$. Hence the characteristic function χ_A of A is a fuzzy left ideal of S.

Conversely, let $y \in A$ and $x \in S$ such that $x \leq y$, this implies that $\chi_A(x) \geq \chi_A(y)$, χ_A being a fuzzy left ideal of S. Then $A = (A]$, by the Proposition 1. Let $xy \in SA$, where $y \in A$, $x \in S$. By definition of a characteristic function $\chi_A(y) = 1$. $\chi_A(xy) \geq \chi_A(y) = 1$, χ_A being a fuzzy left ideal of S. Thus $\chi_A(xy) = 1$, i.e., $xy \in A$. Hence A is a left ideal of S.

Proposition 2: Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$. Then A is an interior ideal of S if and only if the characteristic function χ_A of A is a fuzzy interior ideal of S.

Proof: Suppose A is an interior ideal of S and $a, x, y \in S$ such that $x \leq y$, this implies that $A = (A]$, A being an interior ideal. Then $\chi_A(x) \geq \chi_A(y)$, by the Proposition 1. If $a \notin A$, by definition $\chi_A(a) = 0$. Thus $\chi_A((xa)y) \geq \chi_A(a)$. If $a \in A$, by definition $\chi_A(a) = 1$. $(xa)y \in A$, A being an interior ideal, this implies that $\chi_A((xa)y) = 1$. Thus $\chi_A((xa)y) \geq \chi_A(a)$. Hence the characteristic function χ_A of A is a fuzzy interior ideal of S.

Conversely, let $y \in A$ and $x \in S$ such that $x \leq y$, this implies that $\chi_A(x) \geq \chi_A(y)$, χ_A being a fuzzy interior ideal of S. Then $A = (A]$, by the Proposition 1. Let $t \in (SA)S$, implies $t = (xa)y$, where $a \in A$ and $x, y \in S$. By definition of a characteristic function $\chi_A(a) = 1$. $\chi_A((xa)y) \geq \chi_A(a) = 1$, χ_A being a fuzzy interior ideal of S. Thus $\chi_A((xa)y) = 1$, i.e., $(xa)y \in A$. Hence A is an interior ideal of S.

Lemma 4: Let μ be a fuzzy subset of an ordered AG-groupoid S. Then μ is a fuzzy AG-subgroupoid of S if and only if upper t level $U(\mu; t)$ of μ is an AG-subgroupoid of S for all $t \in (0,1]$.

Proof: S Suppose μ is a fuzzy AG-subgroupoid of S and $x, y \in U(\mu; t)$, this implies that $\mu(x), \mu(y) \geq t$. $\mu(xy) \geq \mu(x) \wedge \mu(y) \geq t$, μ being a fuzzy AG-subgroupoid, i.e., $xy \in U(\mu; t)$. Hence $U(\mu; t)$ is an AG-subgroupoid of S.

Conversely, we have to show that $\mu(xy) \geq \mu(x) \wedge \mu(y)$, $x, y \in S$. We suppose a contradiction $\mu(xy) < \mu(x) \vee \mu(y)$. Assume $\mu(x) = t = \mu(y)$, and this implies that $\mu(x), \mu(y) \geq t$, i.e., $x, y \in U(\mu; t)$. But $\mu(xy) < t$, i.e., $xy \notin U(\mu; t)$, which is a contradiction. Hence $\mu(xy) \geq \mu(x) \wedge \mu(y)$.

Lemma 5: Let μ be a fuzzy subset of an ordered AG-groupoid S. Then μ is a fuzzy left (resp. right) ideal of S if and only if upper t level $U(\mu; t)$ of μ is a left (resp. right) ideal of S for all $t \in (0,1]$.

Proof: Suppose μ is a fuzzy left ideal of S. Let $y \in U(\mu; t)$ and $x \in S$ such that $x \leq y$, this implies that $\mu(y) \geq t$. $\mu(x) \geq \mu(y) \geq t$ and $\mu(xy) \geq \mu(y) \geq t$, μ being a fuzzy left ideal of S. Thus $x, xy \in U(\mu; t)$. Hence $U(\mu; t)$ is a left ideal of S.

Conversely, suppose $U(\mu; t)$ is a left ideal of S and $x, y \in S$ such that $x \leq y$. We have to show that $\mu(x) \geq \mu(y)$ and $\mu(xy) \geq \mu(y)$, we suppose a contradiction $\mu(x) < \mu(y)$ and $\mu(xy) < \mu(y)$. Let $\mu(y) = t$, this implies that $\mu(y) \geq t$, i.e., $y \in U(\mu; t)$. But $\mu(x) < t$ and $\mu(xy) < t$, i.e., $x, xy \notin U(\mu; t)$, which is a contradiction. Hence $\mu(x) \geq \mu(y)$ and $\mu(xy) \geq \mu(y)$.

Proposition 3: Let μ be a fuzzy subset of an ordered AG-groupoid S. Then μ is a fuzzy interior ideal of S if and only if the upper t level $U(\mu;t)$ of μ is an interior ideal of S for all $t \in (0,1]$.

Proof: Suppose μ is a fuzzy interior ideal of S. Let $y \in U(\mu;t)$ and $x \in S$ such that $x \leq y$, this implies that $\mu(y) \geq t$. $\mu(x) \geq \mu(y) \geq t$, μ being a fuzzy interior ideal of S. Thus $\mu(x) \geq t$, i.e., $x \in U(\mu;t)$. Let $a \in U(\mu;t)$ and $x, y \in S$, by definition $\mu(a) \geq t$. $\mu((xa)y) \geq \mu(a) \geq t$, μ being a fuzzy interior ideal of S. Thus $\mu((xa)y) \geq t$, i.e., $(xa)y \in U(\mu;t)$. Hence $U(\mu;t)$ is an interior ideal of S.

Conversely, suppose $U(\mu;t)$ is an interior ideal of S and $x, y, a \in S$ such that $x \leq y$. We have to show that $\mu(x) \geq \mu(y)$, we suppose a contradiction $\mu(x) < \mu(y)$. Let $\mu(y) = t$, this implies that $\mu(y) \geq t$, i.e., $y \in U(\mu;t)$. But $\mu(x) < t$, i.e., $x \notin U(\mu;t)$, which is a contradiction. Hence $\mu(x) \geq \mu(y)$. We have to show that $\mu((xa)y) \geq \mu(a)$, we suppose a contradiction $\mu((xa)y) < \mu(a)$. Let $\mu(a) = t$, this implies that $\mu(a) \geq t$, i.e., $a \in U(\mu;t)$. But $\mu((xa)y) < t$, i.e., $((xa)y) \notin U(\mu;t)$, which is a contradiction. Hence $\mu((xa)y) \geq \mu(a)$.

Lemma 6: Every fuzzy right ideal of an ordered AG-groupoid S with left identity e, is a fuzzy ideal of S.

Proof: Let μ be a fuzzy right ideal of S and $x, y \in S$. Now $\mu(xy) = \mu((ex)y) = \mu((yx)e) \geq \mu(yx) \geq \mu(y)$. Thus μ is a fuzzy ideal of S.

Remark 3: The concept of fuzzy (right, two-sided) ideals coincide in ordered AG-groupoids with left identity e.

Lemma 7: Every fuzzy ideal of an ordered AG-groupoid S is a fuzzy interior ideal of S.

Proof: Let μ be a fuzzy two-sided ideal of S and $x, a, y \in S$. Now, $\mu((xa)y) \geq \mu(xa) \geq \mu(a)$. Hence μ is a fuzzy interior ideal of S.

Proposition 4: Let S be an ordered AG-groupoid with left identity e. Then μ is fuzzy interior ideal if and only if μ is a fuzzy ideal of S.

Proof: Let μ be a fuzzy interior ideal of S and $x, y \in S$. Now $\mu(xy) = \mu((ex)y) \geq \mu(x)$. Thus μ is a fuzzy right ideal of S. Then μ is a fuzzy ideal of S by Lemma 6. The converse is true by Lemma 7.

Lemma 8: Every fuzzy right ideal of a regular ordered AG-groupoid S, is a fuzzy ideal of S.

Proof: Let μ be a fuzzy right ideal of S and $x, y \in S$, this implies that there exists $a \in S$ such that $x \leq (xa)x$. Now, $\mu(xy) \geq \mu(((xa)x)y) = \mu((yx)(xa)) \geq \mu(yx) \geq \mu(y)$. Hence μ is a fuzzy ideal of S.

Remark 4: The concept of fuzzy (right, two-sided) ideals coincide in regular ordered AG-groupoids.

Proposition 5: Let S be a regular ordered AG-groupoid. Then μ is a fuzzy interior ideal if and only if μ is a fuzzy ideal of S.

Proof: Let μ be a fuzzy interior ideal of S and $x, y \in S$, this implies that there exists $a \in S$ such that $x \leq (xa)x$. Now, $\mu(xy) \geq \mu(((xa)x)y) = \mu((yx)(xa)) \geq \mu(x)$. Thus μ is a fuzzy right ideal of S. Then μ is a fuzzy ideal of S by Lemma 8. The converse is true by Lemma 7.

Lemma 9: Every fuzzy right (resp. left) ideal of (2,2) regular ordered AG-groupoid S, is a fuzzy ideal of S.

Proof: Let μ be a fuzzy right ideal of S and $, y \in S$, this implies that there exists $a \in S$ such that $x \leq (x^2a)x^2$. Now $\mu(xy) \geq \mu(((x^2a)x^2)y) = \mu((yx^2)(x^2a)) \geq \mu(yx^2) \geq \mu(y)$. Hence μ is a fuzzy ideal of S. Let μ be a fuzzy left ideal of S. Now $\mu(xy) \geq \mu(((x^2a)x^2)y) = \mu((yx^2)(x^2a)) \geq \mu((xx)a) = \mu((ax)x) \geq \mu(x)$. Hence μ is a fuzzy ideal of S.

Remark 5: The concept of fuzzy (right, left, two-sided) ideals coincide in (2, 2) -regular ordered AG-groupoids.

Proposition 6: Let S be a (2,2) -regular ordered AG-groupoid with left identity e. Then μ is a fuzzy interior ideal if and only if μ is a fuzzy ideal of S.

Proof: Let μ be a fuzzy interior ideal of S and $x, y \in S$, this implies that there exists $a \in S$ such that $x \leq (x^2a)x^2$. Now,

$$\mu(xy) \geq \mu(((x^2a)x^2)y) = \mu((yx^2)(x^2a)) \geq \mu(x^2) = \mu(xx) = \mu((ex)x) \geq \mu(x).$$

Thus μ is a fuzzy right ideal of S. Then μ is a fuzzy ideal of S by Lemma 9. The converse is true by Lemma 7.

Lemma 10: Let S be a right regular ordered AG-groupoid. Then every fuzzy right (resp. left) ideal of S is a fuzzy ideal of S.

Proof: Let μ be a fuzzy right ideal of S and $x, y \in S$, this implies that there exists $a \in S$ such that $x \leq x^2a$. Now,

$$\mu(xy) \geq \mu((x^2a)y) = \mu(((xx)a)y) = \mu(((ax)x)y) = \mu((yx)(ax)) \geq \mu(yx) \geq \mu(y).$$

Hence μ is a fuzzy ideal of S. Let μ be a fuzzy left ideal of S. Now,

$$\mu(xy) \geq \mu((x^2a)y) = \mu(((xx)a)y) = \mu(((ax)x)y) = \mu((yx)(ax)) \geq \mu(ax) \geq \mu(x).$$

Hence μ is a fuzzy ideal of S.

Remark 6: The concept of fuzzy (right, left, two-sided) ideals coincide in right regular ordered AG-groupoids.

Proposition 7: Let S be a right regular ordered AG-groupoid. Then μ is a fuzzy interior ideal if and only if μ is a fuzzy ideal of S.

Proof: Let μ be a fuzzy interior ideal of S and $, y \in S$, this implies that there exists $a \in S$ such that $x \leq x^2 a$. Now $\mu(xy) \geq \mu((x^2 a)y) = \mu(((xx)a)y) = \mu(((ax)x)y) \geq \mu(x)$. Thus μ is a fuzzy right ideal of S. Then μ is a fuzzy ideal of S by Lemma 10. The converse is true by Lemma 7.

Lemma 11: Let S be a left regular ordered AG-groupoid with left identity e. Then every fuzzy right (resp. left) ideal of S is a fuzzy ideal of S.

Proof: Let μ be a fuzzy right ideal of S and $x, y \in S$, that implies that there exists $a \in S$ such that $x \leq ax^2$. Now,

$$\mu(xy) \geq \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y)$$
$$= \mu((y(ax))x) \geq \mu(y(ax)) \geq \mu(y).$$

Hence μ is a fuzzy ideal of S. Let μ be a fuzzy left ideal of S. Now,

$$\mu(xy) \geq \mu((ax^2)y) = \mu((a(xx))y) = \mu((x(ax))y)$$
$$= \mu((y(ax))x) \geq \mu((ax)x) \geq \mu(x).$$

Hence μ is a fuzzy ideal of S.

Remark 7: The concept of fuzzy (right, left, two-sided) ideals coincide in left regular ordered AG-groupoids with left identity e.

Proposition 8: Let S be a left regular ordered AG-groupoid with left identity e. Then μ is a fuzzy interior ideal if and only if μ is a fuzzy ideal of S.

Proof: Let μ be a fuzzy interior ideal of S and $x, y \in S$, this implies that there exists $a \in S$ such that $x \leq ax^2$. Now,

$$\mu(xy) \geq \mu((ax^2)y) = \mu((a(xx))y)$$
$$= \mu((x(ax))y) = \mu(((ex)(ax))y)$$
$$= \mu(((xx)(ae))y) = \mu((((ae)x)x)y) \geq \mu(x).$$

Thus μ is a fuzzy right ideal of S. Hence μ is a fuzzy ideal of S by Lemma 11. The converse is true by Lemma 7.

Theorem 1: Let S be a right regular locally associative ordered AG-groupoid with left identity e. Then for every fuzzy interior ideal μ of S, $\mu(a^n) = \mu(a^{2n})$, where n is any positive integer, for all $a \in S$.

Proof: For $n = 1$. Let $a \in S$, this implies that there exists $x \in S$ such that $a \leq a^2 x$. Now $\mu(a) \geq \mu(a^2 x) = \mu((ea^2)x) \geq \mu(a^2) \geq min\{\mu(a), \mu(a)\} = \mu(a)$, ($\mu$ is a fuzzy ideal of S by Proposition 7). Hence $\mu(a) = \mu(a^2)$. Now $a^2 = aa \leq (a^2 x)(a^2 x) = a^4 x^2$, then the result is true for $n = 2$. Suppose result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{2k})$. Now,

$$a^{k+1} = a^k a \leq (a^{2k} x^k)(a^2 x) = a^{2(k+1)} x^{(k+1)}.$$
$$\mu(a^{k+1}) \geq \mu(a^{2(k+1)} x^{(k+1)}) = \mu((ea^{2(k+1)}) x^{(k+1)})$$
$$\geq \mu(a^{2(k+1)}) = \mu(a^{2k+2}) = \mu(a^{k+1} a^{k+1})$$
$$\geq min\{\mu(a^{k+1}), \mu(a^{k+1})\} = \mu(a^{k+1}).$$

Thus $\mu(a^{k+1}) = \mu(a^{2(k+1)})$. Hence by the induction method, the result is true for all positive integers.

Lemma 12: Let S be a right regular locally associative ordered AG-groupoid with left identity e. Then for every fuzzy interior ideal μ of S, $\mu(ab) = \mu(ba)$ for all $a, b \in S$.

Proof: Let $a, b \in S$. By using **Theorem** (for $n = 1$). Now,

$$\mu(ab) = \mu((ab)^2) = \mu((ab)(ab))$$
$$= \mu((ba)(ba)) = \mu((ba)^2) = \mu(ba).$$

Theorem 2: Let S be a regular and right regular locally associative ordered AG-groupoid with left identity e. Then for every fuzzy interior ideal μ of S, $\mu(a^n) = \mu(a^{3n})$, where n is any positive integer, for all $a \in S$.

Proof: For $n = 1$. Let $a \in S$, this implies that there exists $x \in S$ such that $a \leq (ax)a$ and $a \leq a^2 x$. Now $a \leq (ax)a \leq (ax)(a^2 x) = a^3 x^2$. Thus,

$$\mu(a) \geq \mu(a^3 x^2) = \mu((ea^3)x^2) \geq \mu(a^3)$$
$$= \mu(aa^2) \geq min\{\mu(a), \mu(a^2)\}$$
$$\geq min\{\mu(a), \mu(a), \mu(a)\} = \mu(a).$$

Hence $\mu(a) = \mu(a^3)$. Now $a^2 = aa \leq (a^3 x^2)(a^3 x^2) = a^6 x^4$, then the result is true for $n = 2$. Suppose result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{3k})$. Now,

$$a^{k+1} = a^k a \leq (a^{3k} x^{2k})(a^3 x^2) = a^{3(k+1)} x^{2(k+1)}.$$
$$\mu(a^{k+1}) \geq \mu(a^{3(k+1)} x^{2(k+1)}) = \mu((ea^{3(k+1)}) x^{2(k+1)})$$
$$\geq \mu(a^{3(k+1)})$$
$$= \mu(a^{3k+3}) = \mu(a^{k+1} a^{2k+2}) \geq min\{\mu(a^{k+1}), \mu(a^{2k+2})\}$$
$$\geq min\{\mu(a^{k+1}), \mu(a^{k+1}), \mu(a^{k+1})\} = \mu(a^{k+1}).$$

Thus $\mu(a^{k+1}) = \mu(a^{3(k+1)})$. Hence by the induction method, the result is true for all positive integers.

Lemma 13: Let S be a weakly regular ordered AG-groupoid. Then every fuzzy right (resp. left) ideal is a fuzzy ideal of S.

Proof: Let μ be a fuzzy right ideal of S and $x, y \in S$, this implies that there exist $a, b \in S$ such that $x \leq (xa)(xb)$. Now,

$$\mu(xy) \geq \mu(((xa)(xb))y) = \mu((((xb)a)x)y)$$
$$= \mu((((ab)x)x)y) = \mu((yx)((ab)x))$$
$$= \mu((yx)(nx)) \; say \; ab = n$$
$$\geq \mu(yx) \geq \mu(y).$$

Hence μ is a fuzzy ideal of S. Let μ be a fuzzy left ideal of S. Now,

$\mu(xy) \geq \mu(((xa)(xb))y) = \mu((((xb)a)x)y)$
$= \mu((((ab)x)x)y) = \mu((yx)((ab)x))$
$= \mu((yx)(nx))$ $say\ ab = n$
$\geq \mu(nx) \geq \mu(x)$.

Hence μ is a fuzzy ideal of S.

Remark 8: The concept of fuzzy (right, left, two-sided) ideals coincide in weakly regular ordered AG-groupoids.

Proposition 9: Let S be a weakly regular ordered AG-groupoid. Then μ is a fuzzy interior ideal if and only if μ is a fuzzy ideal of S.

Proof: Let μ be a fuzzy interior ideal of S and $x, y \in S$, this implies that there exist $a, b \in S$ such that $x \leq (xa)(xb)$. Now $\mu(xy) \geq \mu(((xa)(xb))y) = \mu((((xb)a)x)y) \geq \mu(x)$. Thus μ is a fuzzy right ideal of S. Hence μ is a fuzzy ideal of S by Lemma 13. The converse is true by Lemma 7.

Proposition 10: Let S be an ordered AG-groupoid with left identity e. Then S is a weakly regular if and only if S is completely regular.

Proof: Suppose S is a weakly regular ordered AG-groupoid. Let $a \in S$, then there exist $x, y \in S$ such that $a \leq (ax)(ay)$. Now $a \leq (ax)(ay) = (aa)(xy) = a^2t$, for some $t \in S$, this implies that $a \leq a^2t$. Thus S is a right regular ordered AG-groupoid.

Now $a \leq (ax)(ay) = (yx)(aa) = ta^2$, for some $t \in S$, this imply that $a \leq ta^2$. Thus S is a left regular ordered AG-groupoid. Now,

$a \leq (ax)(ay) = (aa)(xy) = a^2t = (aa)t = (ta)a$
$\leq (t(ta^2))a = (t(t(aa)))a = (t(a(ta)))a$
$= (a(t(ta)))a = (as)a$, $say\ t(ta) = n$.

This implies that $a \leq (as)a$, for some $s \in S$. Thus S is a regular ordered AG-groupoid. Hence S is a completely regular ordered AG-groupoid.

Conversely, let S be a completely regular ordered AG-groupoid. Let $a \in S$, and then there exists $x \in S$ such that $a \leq (ax)a$, $a \leq a^2x$ and $a \leq xa^2$. Now,

$a \leq (ax)a \leq (ax)(xa^2) = (ax)(x(aa))$
$= (ax)(a(xa)) = (ax)(ay)$, $say\ xa = y$.

This implies that $a \leq (ax)(ay)$, for some $x, y \in S$. Hence S is weakly regular ordered AG-groupoid.

Lemma 14: Every fuzzy right ideal of an intra-regular ordered AG-groupoid S, is a fuzzy ideal of S.

Proof: Let μ be a fuzzy right ideal of S and $x, y \in S$, this implies that there exist $a, b \in S$ such that $x \leq (ax^2)b$. Now,
$\mu(xy) \geq \mu(((ax^2)b)y) = \mu((yb)(ax^2)) \geq \mu(yb) \geq \mu(y)$.

Hence μ is a fuzzy ideal of S.

Remark 9: The concept of fuzzy (right, two-sided) ideals coincide in intra-regular ordered AG-groupoids.

Proposition 11: Let S be an intra-regular ordered AG-groupoid with left identity . Then μ is a fuzzy interior ideal if and only if μ is a fuzzy ideal of S.

Proof: Let μ be a fuzzy interior ideal of S and $x, y \in S$, this implies that there exist $a, b \in S$ such that $x \leq (ax^2)b$. Now,

$xy \leq ((ax^2)b)y = (yb)(ax^2) = n(a(xx))$
$= n(x(ax))$, $say\ yb = n$
$= (en)(x(ax)) = (ex)(n(ax)) = (ex)m$, $say\ n(ax) = m$.

Thus $\mu(xy) \geq \mu((ex)m) \geq \mu(x)$. Hence μ is a fuzzy ideal of S. The converse is true by Lemma 7.

Theorem 3: L Let S be an intra-regular locally associative ordered AG-groupoid. Then for every fuzzy interior ideal μ of S, $\mu(a^n) = \mu(a^{2n})$, where n is any positive integer, for all $a \in S$.

Proof: For $n = 1$. Let $\in S$, this implies that there exist $x, y \in S$ such that $a \leq (xa^2)y$. Now $\mu(a) \geq \mu((xa^2)y) \geq \mu(a^2) = \mu(aa) \geq min\{\mu(a), \mu(a)\} = \mu(a)$, ($\mu$ is a fuzzy ideal of S by Proposition 11. Hence,

$\mu(a) = \mu(a^2)$.

Now,

$a^2 = aa \leq ((xa^2)y)((xa^2)y) = ((xa^2)(xa^2))y^2 = (x^2a^4)y^2$,

then the result is true for $n = 2$. Suppose that result is true for $n = k$, i.e., $\mu(a^k) = \mu(a^{2k})$. Now,

$a^{k+1} = a^k a\left((x^k a^{2k})y^k\right)\left((x^2a)y\right) = (x^{k+1}a^{2(k+1)})y^{k+1}$
$\mu(a^{k+1}) \geq \mu((x^{k+1}a^{2(k+1)})y^{k+1}) \geq \mu(a^{2(k+1)})$
$= \mu(a^{(k+1)}a^{(k+1)})$
$\geq min\{\mu(a^{(k+1)}), \mu(a^{(k+1)})\} = \mu(a^{(k+1)})$.

Thus $\mu(a^{k+1}) = \mu(a^{2(k+1)})$. Hence by the induction method, the result is true for all positive integers.

Lemma 15: Let S be an intra-regular locally associative ordered AG-groupoid with left identity e. Then for every fuzzy interior ideal μ of S. $\mu(ab) = \mu(ba)$ for all $a, b \in S$.

Proof: Same as Lemma 12.

4. Conclusion

Our ambition is to inspire the study and maturity of non-associative algebraic structure (ordered AG-groupoid). The objective is to explain original methodological developments on ordered AG-groupoids, which will be very helpful for the upcoming theory of algebraic structure. The ideal of a fuzzy set to the characterizations of ordered semigroups is captivating the great attention of algebraists. The aim of this paper is to investigate, the study of (regular, left regular, right regular, (2, 2)-regular, left weakly regular, right weakly regular, intra-regular) ordered AG-groupoids by using of the fuzzy left (right, interior) ideals.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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