On effects of viscous damping of harmonically varying axially moving string

Sanaullah Dehraj *, Rajab A. Malookani, Sajad H. Sandilo

Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Sciences and Technology, Nawabshah, Pakistan

A B S T R A C T

In this study, the stability of an axially moving string under the influence of viscous damping in terms of transverse vibrations has been examined. Mathematically, an axially moving string can be expressed as a linear-homogenous partial differential equation with the initial and boundary conditions. The axial speed of string is taken to be time-varying, sinusoidal, and small compared to wave velocity. In order to approximate the exact solutions of the initial-boundary value problem, a Fourier-expansion method, together with the two timescales perturbation method, has been used. General resonance case and the detuning case have been studied in detail. The total energy of an infinite-dimensional coupled system has been obtained. Under certain values of the damping parameter, this total mechanical energy is either obtained to be bound or unbound. In addition, it turned out that there is a possibility of mode-truncation depending on the certain values of the damping parameter.

1. Introduction

Engineering devices, such as conveyor belts, elevator cable, aerial-cable, tramways, power transmission belts, plastic films, oil pipelines, magnetic and paper tapes, band-saw blades, crane and mining hoists are considered in the class of axially moving continuous systems. These axially moving systems can be used in diverse engineering applications such as civil, mechanical, and aerospace engineering. Their applications are limited due to vibrations, specifically, the transverse vibrations. The understanding of transversal vibrations of axially moving continua is important for the design and manufacture of these systems. The vibrations are caused due to many reasons such as winds, storms, earthquakes, irregular speed and eccentricity of the driven motor, and non-uniformity in the material properties. All of these diverse factors contribute to one or the other way to cause the vibrations, which can lead to severe damage or the failure of these systems. To control the vibrations in these mechanical systems, differently designed

dampers can be used. Both the theorists and experimentalists have studied the transverse vibrations under the effect of damping, either present in the system (Krenk, 2000; Marynowski and Kapitaniak, 2007; Shahruz, 2009; Rossikhin and Shitikova, 2013) or at the boundary (Darmawijoyo and van Horssen, 2002; Gaiko and van Horssen, 2015; Malookani et al., 2019; Akkaya and van Horssen, 2019). Wickert and Mote (1990) carried out the modal analysis to examine the dynamics of string-like and beam-like models. The authors obtained the eigenfunctions for stationary string and beam equations. The dynamics of the string-like model are examined by using discretization and then by using two timescales perturbation technique in a study by Pakdemirli and Ulsoy (1997) and a transition behavior of string-like equation to the beam-like equation is studied by Oz et al. (1998). A thorough review of the model describing the transverse vibration of an axially moving string system is given in Chen (2005). The free nonlinear vibrations of the viscoelastic string system are examined via the application of two timescales perturbation method by Zhang and Zu (1998). The weakly nonlinear string-like equation is studied by keeping one end as fixed, and the dashpot system at the other end in the two timescales perturbation method was used to compute the amplitude response of the system (Darmawijoyo et al., 2003). It has been observed that the solution tends to zero by an increasing amount of damping. Sandilo and van

Horssen (2012) studied the tensioned Euler-Bernoulli beam equation by considering the spring-dashpot system at one end, whereas the other end was simply supported. A two timescales perturbation method was employed, and authors did not draw any satisfactory conclusion regarding whether the energy increases, decreases, or remains conserved. Gaiko and van Horssen (2015) studied the equations of motion with non-classical boundary conditions. A two timescales perturbation method was employed to find an approximate analytic solution. It was observed that the amplitude of transverse vibrations reduces substantially by increasing the damping factor. Sandilo et al. (2016) studied the initial-boundary value problem for the damped axially translating continuum. A two timescales perturbation method was employed to obtain analytic solutions. The authors depicted that the oscillation modes are damped and damping rates depending on the mode numbers k, which was indeed expected. Akkaya and van Horssen (2017) used the Laplace transform method and examined the vibrations of the beam equation on the semi-infinite interval. It was shown that the numerical results, in fact, reach the analytic results for a sufficiently larger length and a larger number of modes. Malookani and van Horssen (2015) studied the string-like model with the fixed boundary conditions. They utilized a two timescales perturbation method with a conjunction of Laplace transform method in search of the infinite-mode approximate solutions. They have shown that Galerkin's truncation method is not applicable in order to obtain approximations valid on long timescales. However, their study was restricted to the un-damped system.

This paper aims to examine the (in) stability of the axially moving string under the influence of viscous damping for general resonance and the detuning case. The transversal vibrations of the system are modeled as a second-order homogeneous linear partial differential equation with variable coefficients. The Fourier-sine series, with a conjunction of a two timescales perturbation method, are used to determine the approximate-analytic solutions.

This paper is structured as follows. In section 2 the governing equations of motion are established. In section 3 the analytic approximations of the exact solutions of the initial-boundary value problem are constructed by using the Fourier expansion method with a conjunction of a two timescales perturbation method. A mathematical analysis of a coupled system of ODEs and a Galerkin's truncation method will be applied to truncate few modes from the infinite-dimensional system of ODEs. The detuning case will also be discussed and analyzed in detail. In section 4, the results and the discussions are presented. Finally, some conclusions will be drawn in section 5.

2. Mathematical model

An axially translating string under the effect of small viscous damping δ_0, moving with non-constant velocity $V(t)$ is represented in Fig. 1. There is no vertical displacement at the end point of string, i.e. $u = 0$ at $x = 0$ and $x = L$, and pair of pulleys are apart by constant L. The damped string-like equation is formulated by extended Hamilton's principle (Maitlo et al., 2016). Thus, the following IBVP is considered as under,

$$\rho(u_{tt} + 2Vu_{xt} + V_t u_x + V^2 u_{xx}) - Pu_{xx} + \delta_0(u_t + Vu_x) = 0, t \geq 0, 0 < x < L, \tag{1}$$
$$u(0,t) = u(L,t) = 0, \quad t \geq 0, \tag{2}$$
$$u(x,0) = \psi(x), \quad \text{and} \quad u_t(x,0) = \phi(x), \ 0 < x < L, \tag{3}$$

where t is the time, x be the axial coordinate in the horizontal direction, time-dependent belt velocity is represented by V, transversal displacement of the string is denoted by $u(x,t)$ and linear constant mass-density per unit length is denoted by ρ, P is the constant tension and δ_0 is small viscous damping coefficient. Displacement and the velocity at time $t = 0$ are represented by the functions $\psi(x)$ and $\phi(x)$, respectively.
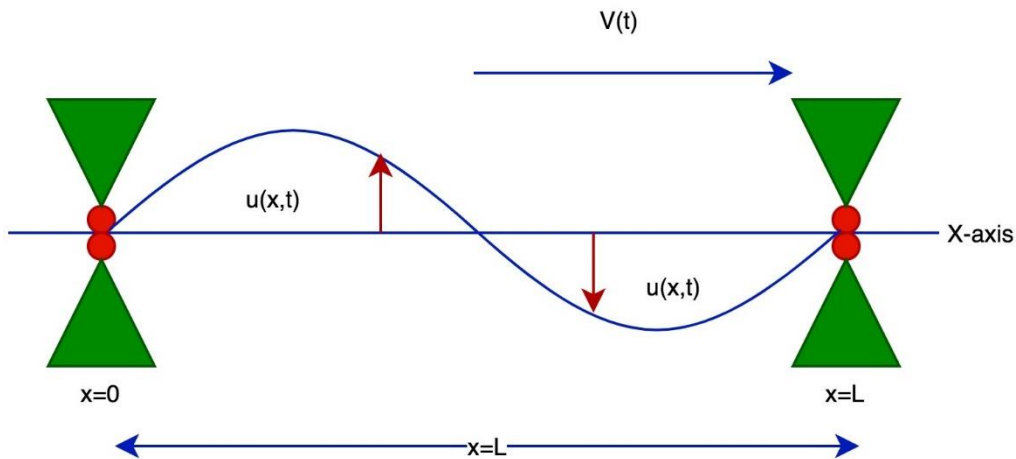

Fig. 1: The schematic model of a damped axially moving belt

In order to write IBVP (Eqs. 1-3) in dimensionless form, we use the following assumptions:

$$x^* = \frac{x}{L}, V^* = \frac{V}{c}, t^* = \frac{ct}{L}, u^*(x^*, t^*) = \frac{u(x,t)}{L}, \Omega^* = \frac{L\Omega}{c}, \psi^*(x) = \frac{\psi(x)}{L},$$
$$\phi^*(x^*) = \frac{\phi(x)}{c}, \delta_0^* = \frac{\delta_0 L}{\rho c} \tag{4}$$

By substitution Eq. 4 into the equations of motion (Eqs. 1-3) yields the following equations (asterisk are ignored)

$$u_{tt} + 2Vu_{xt} + V_t u_x + (V^2 - 1)u_{xx} + \delta_0(u_t + Vu_x) = 0, \quad t \geq 0, \quad 0 < x < 1, \tag{5}$$

with the Dirichlet boundary conditions:

$$u(0, t) = u(1, t) = 0, \quad t \geq 0, \tag{6}$$

and the general initial conditions:

$$u(x, 0) = \psi(x), \text{ and } u_t(x, 0) = \phi(x), \quad 0 < x < 1. \tag{7}$$

The axial velocity of the belt varies with time due to many reasons such as the eccentricity of pulleys, non-uniformity of material properties, irregular speed of the driven motor, so in this study, we have considered harmonically low mean time-varying belt velocity as under,

$$V(t) = \varepsilon(V_0 + \alpha \sin(\Omega t)). \tag{8}$$

To solve IBVP (Eqs. 5-7) the following assumptions are considered:

(i) The transversal vibration $u(x, t)$ is taken into account only.
(ii) The dimensional quantity $\delta_0 L$ is small in comparison to ρc, where $c = \sqrt{\frac{P}{\rho}}$. Thus, it is reasonable to express:

$$\delta_0^* = \frac{\delta_0 L}{\rho c} = \varepsilon\delta \tag{9}$$

(iii) The axial velocity $V(t)$ of the belt is small in comparison to wave-velocity c. Thus, we can express $|V| \ll c$.
(iv) The belt always moves in a forward direction only, so it is necessary to impose a condition $V_0 > |\alpha|$.
(v) Bending stiffness, external excitations, and effect of gravity are neglected.
(vi) $\alpha, V_0, \delta_0, \rho, P, \Omega, c$ are the non-zero positive constants.

3. The analytic approximations

This section discusses the solution of the initial-boundary value problem (Eqs. 5-7). By plugging Eq. 8 and Eq. 9 into Eq. 5, one obtains:

$$u_{tt} - u_{xx} = -\varepsilon[2(V_0 + \alpha\sin(\Omega t))u_{xt} + \alpha\Omega\cos(\Omega t)u_x + \delta u_t] + O(\varepsilon^2) \tag{10}$$

with the Dirichlet boundary conditions:

$$u(0, t; \varepsilon) = u(1, t; \varepsilon) = 0, \quad t \geq 0 \tag{11}$$

and the initial conditions:

$$u(x, 0; \varepsilon) = \psi(x), \text{ and } u_t(x, 0; \varepsilon) = \phi(x), \quad 0 < x < 1. \tag{12}$$

The boundary conditions (Eq. 11) suggest that the solution of Eq. 10 can be expanded in terms of Fourier-sine series given as under:

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t; \varepsilon)\sin(n\pi x). \tag{13}$$

The following orthogonally conditions holds:

$$\int_0^1 \sin(n\pi x)\sin(k\pi x)\, dx = \begin{cases} 0, & \text{for } n \neq k \\ \frac{1}{2}, & \text{for } n = k \end{cases} \tag{14}$$

$$\int_0^1 \cos(n\pi x)\sin(k\pi x)\, dx = \begin{cases} 0, & \text{for } (n \pm k) \text{ even} \\ -\frac{2k}{(n^2 - k^2)\pi}, & \text{for } (n \pm k) \text{ odd} \end{cases}. \tag{15}$$

By substitution Eq. 13 and all its required derivatives into Eq. 10, it follows that:

$$\sum_{n=1}^{\infty}(\ddot{u}_n + (n\pi)^2 u_n)\sin(n\pi x) = -\varepsilon\sum_{n=1}^{\infty}(n\pi)[2(V_0 + \alpha\sin(\Omega t)\dot{u}_n + \alpha\Omega\cos(\Omega t)u_n)]\cos(n\pi x) - \varepsilon\delta\sum_{n=1}^{\infty}\dot{u}_n\sin(n\pi x) + O(\varepsilon^2). \tag{16}$$

Multiply Eq. 16 with $\sin(k\pi x)$ both sides and then integrates w.r.t to x over the interval [0 1], and using Eq. 14 and Eq. 15 it yields:

$$(\ddot{u}_k + (k\pi)^2 u_k) = \varepsilon\sum_{\substack{n=1 \\ n\pm k \text{ is odd}}}^{\infty}\left(\frac{nk}{n^2-k^2}\right)[8(V_0 + \alpha\sin(\Omega t))\dot{u}_n + 4\alpha\Omega\cos(\Omega t)u_n] - \varepsilon\delta\dot{u}_k + O(\varepsilon^2). \tag{17}$$

Eq. 17 represents a system of infinite ODE's and is not easy to solve in terms of elementary functions. We will solve Eq. 17 for different values of fluctuation parameter Ω by using a perturbation method. Let the solution of Eq. 17 is in the form $u_k(t; \varepsilon) = v_k(t_0, t_1; \varepsilon)$, where $t_0 = t$ (fast timescale) and $t_1 = \varepsilon t$ (slow timescales).

We use the following transformations for the time derivatives in terms of the timescales t_0, and t_1:

$$\frac{du_k}{dt} = \frac{\partial v_k}{\partial t_0} + \varepsilon\frac{\partial v_k}{\partial t_1} \tag{18}$$

$$\frac{d^2 u_k}{dt^2} = \frac{\partial^2 v_k}{\partial t_0^2} + 2\varepsilon\frac{\partial^2 v_k}{\partial t_0 \partial t_1} + \varepsilon^2\frac{\partial^2 v_k}{\partial t_1^2}. \tag{19}$$

Substitution of Eq. 18 and Eq. 19 into Eq. 17 it yields,

$$\frac{\partial^2 v_k}{\partial t_0^2} + 2\varepsilon\frac{\partial^2 v_k}{\partial t_0 \partial t_1} + (k\pi)^2 v_k = \varepsilon\sum_{\substack{n=1 \\ n\pm k \text{ is odd}}}^{\infty}\left(\frac{nk}{n^2-k^2}\right)\left[4\alpha\Omega\cos(\Omega t_0)v_n + 8(V_0 + \alpha\sin(\Omega t_0))\frac{\partial v_n}{\partial t_0}\right] - \delta\varepsilon\frac{\partial v_k}{\partial t_0} + O(\varepsilon^2). \tag{20}$$

An approximation of $v_k(t_0, t_1; \varepsilon)$ can be extended as under,

$$v_k(t_0, t_1; \varepsilon) = v_{k0}(t_0, t_1) + \varepsilon v_{k1}(t_0, t_1) + \cdots \qquad (21)$$

Plugging of Eq. 21 into Eq. 20 and then collecting the coefficients of ε^0 and ε on both sides, we obtain $O(1)$ and $O(\varepsilon)$-problem for v_{k0} and v_{k1} as follows:

$$O(1): \frac{\partial^2 v_{k0}}{\partial t_0^2} + (k\pi)^2 v_{k0} = 0 \qquad (22)$$

$$O(\varepsilon): \frac{\partial^2 v_{k1}}{\partial t_0^2} + (k\pi)^2 v_{k1} = -2\frac{\partial^2 v_{k0}}{\partial t_0 \partial t_1} - \delta\frac{\partial v_{k0}}{\partial t_0} +$$
$$\sum_{\substack{n=1 \\ n\pm k \text{ is odd}}}^{\infty} \left(\frac{nk}{n^2-k^2}\right)\left[4\alpha\Omega\cos(\Omega t_0)v_{n0} + 8(V_0 + \alpha\sin(\Omega t_0))\frac{\partial v_{n0}}{\partial t_0}\right]. \qquad (23)$$

We use direct integration to solve the $O(1)$-problem:

$$v_{k0}(t_0, t_1) = A_{k0}(t_1)\cos(k\pi t_0) + B_{k0}(t_1)\sin(k\pi t_0) \qquad (24)$$

where $A_{k0}(t_1)$ and $B_{k0}(t_1)$ are still arbitrary functions, and to be determined to make the $O(\varepsilon)$-problem free of unbounded terms.

3.1. General resonance case $\Omega = m\pi$

In this section, the general resonance case that is, when the frequency Ω is equal to m^{th} times the natural frequency of the damped axially moving string, i.e., $\Omega = m\pi$, where $m = 1,3,5\cdots$. By setting $\Omega = m\pi$ into $O(\varepsilon)$-problem (23), and in order to avoid the secular terms in $v_{k1}(t_0, t_1)$; $A_{k0}(t_1)$ and $B_{k0}(t_1)$ has to satisfy the following solvability conditions:

$$\frac{dA_{k0}}{d\bar{t}_1} = -\frac{\bar{\delta}}{2}A_{k0}(t_1) + \left[(k+m)B_{(k+m)0} + (k-m)B_{(k-m)0} - (m-k)B_{(m-k)0}\right] \qquad (25)$$
$$\frac{dB_{k0}}{d\bar{t}_1} = -\frac{\bar{\delta}}{2}B_{k0}(t_1) - \left[(k+m)A_{(k+m)0} + (k-m)A_{(k-m)0} + (m-k)A_{(m-k)0}\right]$$

where, $\bar{\delta} = \frac{\delta m}{\alpha}, \bar{t}_1 = \frac{\alpha t_1}{m}$ $k = 1,2,3,\ldots$ and the arbitrary functions A_{k0} and B_{k0} are defined to be zero for non-positive indices k. For simplicity, we will drop the bar from $\bar{t}_1$ and $\bar{\delta}$. The coupled system (Eq. 25) of ODEs represents that there are infinitely many vibrations modes in the system. In the following subsection, we will analyze the coupled system (Eq. 25) by using Galerkin's truncation method.

3.1.1. Galerkin's truncation method

In this subsection, we will study the coupled system (Eq. 25) by truncating it to a finite number of modes. This study will compute the eigenvalues for $m = 3$ and $m = 5$ up to 10 vibration modes. In order to compute the eigenvalues for the first four modes of vibration for $= 3$, the coupled system (Eq. 25) yields,

$$\dot{X} = AX \qquad (26)$$

where

$$X = \begin{bmatrix} A_{10} \\ B_{10} \\ A_{20} \\ B_{20} \\ A_{30} \\ B_{30} \\ A_{40} \\ B_{40} \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{\delta}{2} & 0 & 0 & -2 & 0 & 0 & 0 & 4 \\ 0 & -\frac{\delta}{2} & -2 & 0 & 0 & 0 & -4 & 0 \\ 0 & -1 & -\frac{\delta}{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -\frac{\delta}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\delta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\delta}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\frac{\delta}{2} & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\delta}{2} \end{bmatrix}. \qquad (27)$$

The eigenvalues of matrix given in Eq. 27 are $-\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2} \pm \sqrt{2}i, -\frac{\delta}{2} \pm \sqrt{2}i$ which are a multiplicity of two and are clearly damped out. By using the Maple16, the eigenvalues of the coupled system (Eq. 25) have been computed up to 10 modes of vibrations for $m = 3$ and $m = 5$ and are presented in Table 1 and Table 2, respectively. For $\delta = 0$ we get similar eigenvalues as obtained by Malookani and van Horssen (2015).

Table 1: Eigenvalues of coupled system (Eq. 25) for $m = 3$

No. of Modes	Eigenvalues of A	Order of A
1	$-\frac{\delta}{2}$	2
2	$-\frac{\delta}{2} \pm \sqrt{2}$	4
3	$-\frac{\delta}{2}, -\frac{\delta}{2} \pm \sqrt{2}$	6
4	$-\frac{\delta}{2} \pm \sqrt{2}i, -\frac{\delta}{2}, -\frac{\delta}{2}$	8
5	$-\frac{\delta}{2}, -\frac{\delta}{2} \pm 0.40 + 2.48i, -\frac{\delta}{2} \pm 0.40 - 2.48i$	10
6	$-\frac{\delta}{2} \pm 4.24i, -\frac{\delta}{2} \pm 0.40 + 2.48i, -\frac{\delta}{2} \pm 0.40 - 2.48i$	12
7	$-\frac{\delta}{2}, -\frac{\delta}{2} \pm 4.24i, -\frac{\delta}{2} \pm 2.89i, -\frac{\delta}{2} \pm 5.64i$	14
8	$-\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2} \pm 4.24i, -\frac{\delta}{2} \pm 2.89i, -\frac{\delta}{2} \pm 5.64i$	16
9	$-\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2} \pm 2.89i, -\frac{\delta}{2} \pm 5.64i, -\frac{\delta}{2} \pm 8.48i$	18
10	$-\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2} \pm 2.08i, -\frac{\delta}{2} \pm 2.55i, -\frac{\delta}{2} \pm 8.48i, -\frac{\delta}{2} \pm 9.96i$	20

It can be observed from Table 1 that, 1st, 4th, 7th, 8th, 9th, and 10th mode-amplitudes are stable in nature and 1st, 2nd, and 7th are mode-amplitudes are stable as shown in Table 2; however remaining modes in both tables are stable at certain values of damping parameter.

3.1.2. Mathematical analysis of the coupled system (Eq. 25)

In this subsection, we will analyze the coupled system of ODEs given in Eq. 25 to compute the

energy for a damped axially moving string for general resonance case as follows:

For the transformations $X_{k0}(t_1) = kA_{k0}(t_1)$ and $Y_{k0}(t_1) = kB_{k0}(t_1)$, coupled system (25) yields,

$$\begin{cases} X_{k0}\dot{X}_{k0} = -\frac{\delta}{2}X_{k0}^2 + k\left[X_{k0}Y_{(k+m)0} + X_{k0}Y_{(k-m)0} - X_{k0}Y_{(m-k)0}\right] \\ Y_{k0}\dot{Y}_{k0} = -\frac{\delta}{2}Y_{k0}^2 - k\left[Y_{k0}X_{(k+m)0} + Y_{k0}X_{(k-m)0} + Y_{k0}X_{(m-k)0}\right] \end{cases} \tag{29}$$

Table 2: Eigenvalues of coupled system (Eq. 25) for $m = 5$

No. of Modes	Eigenvalues of A	Order of A
1	$-\frac{\delta}{2}$	2
2	$-\frac{\delta}{2}, -\frac{\delta}{2}$	4
3	$-\frac{\delta}{2} \pm \sqrt{6}, -\frac{\delta}{2}$	6
4	$-\frac{\delta}{2} \pm \sqrt{6}, -\frac{\delta}{2} \pm 2$	8
5	$-\frac{\delta}{2}, -\frac{\delta}{2} \pm \sqrt{6}, -\frac{\delta}{2} \pm 2$	10
6	$-\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2} \pm \sqrt{6}, -\frac{\delta}{2} \pm \sqrt{2}i$	12
7	$-\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2} \pm \sqrt{2}i, -\frac{\delta}{2} \pm 2\sqrt{2}i$	14
8	$-\frac{\delta}{2}, -\frac{\delta}{2}, -\frac{\delta}{2} \pm \sqrt{2}i, -\frac{\delta}{2} \pm 1.08 + 4.14i, -\frac{\delta}{2} \pm 1.08 - 4.14i$	16
9	$-\frac{\delta}{2}, -\frac{\delta}{2} \pm 2.64i, -\frac{\delta}{2} \pm 5.57i, -\frac{\delta}{2} \pm 1.08 + 4.14i, -\frac{\delta}{2} \pm 1.08 - 4.14i$	18
10	$-\frac{\delta}{2} \pm 7.07i, -\frac{\delta}{2} \pm 2.64i, -\frac{\delta}{2} \pm 5.57i, -\frac{\delta}{2} \pm 1.08 + 4.14i, -\frac{\delta}{2} \pm 1.08 - 4.14i$	20

By adding both the equations in Eq. 29 on both sides and by taking the sum from $k = 1$ to ∞, it yields,

$$\frac{1}{2}\frac{d}{dt_1}\sum_{k=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) = -\frac{\delta}{2}\sum_{K=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) + m\sum_{K=1}^{\infty}(X_{(k+m)0}Y_{k0} - X_{k0}Y_{(k+m)0}) + (1)(-X_{(m-1)0}Y_{10} - X_{10}Y_{(m-1)0}) + (2)(-X_{(m-2)0}Y_{20} - X_{20}Y_{(m-2)0}) + \cdots + (m-2)(-X_{20}Y_{(m-2)0} - X_{(m-2)0}Y_{20}) + (m-1)(-X_{10}Y_{(m-1)0} - X_{(m-1)0}Y_{10}). \tag{30}$$

Differentiate Eq. 30 both sides with respect to, t_1, we obtain,

$$\frac{1}{2}\frac{d^2}{dt_1^2}\sum_{K=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) + \delta\frac{d}{dt_1}\sum_{K=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) + \left(\frac{\delta^2}{2} - 2m^2\right)\sum_{K=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) = 0 \tag{31}$$

and then by putting $\sum_{K=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) = w(t_1)$ into Eq. 31 yields,

$$\frac{d^2w(t_1)}{dt_1^2} + 2\delta\frac{dw(t_1)}{dt_1} + (\delta^2 - 4m^2)w(t_1) = 0. \tag{32}$$

The solution of Eq. (32) is,

$$w(t_1) = C_1 e^{(-\delta+2m)t_1} + C_2 e^{(-\delta-2m)t_1} \tag{33}$$

where C_1 and C_2 are non-zero constants and can be obtained by using the initial conditions of the

$$\begin{cases} \frac{dX_{k0}}{dt_1} = -\frac{\delta}{2}X_{k0} + k\left[Y_{(k+m)0} + Y_{(k-m)0} - Y_{(m-k)0}\right] \\ \frac{dY_{k0}}{dt_1} = -\frac{\delta}{2}Y_{k0} - k\left[X_{(k+m)0} + X_{(k-m)0} + X_{(m-k)0}\right] \end{cases} \tag{28}$$

where, $k = 1,2,3, \ldots$ and the functions $X_{k0}(t_1) = 0$, $Y_{k0}(t_1) = 0$, $\forall k \leq 0$. Thus it yields,

problem. Thus the following cases arise for damping parameter:

Case 1: For $\delta = 2m$ the energy of the system first decreases and then becomes constant, so in this case, the system is stable.

Case 2: For $\delta > 2m$, the energy of the system will tend to zero for sufficiently large time, so the system remains stable.

Case 3: For $\delta < 2m$, the energy of the system will grow exponentially as time increases, so the system is unstable.

when $\delta = 0$ we will get the same system as obtained in a study by Malookani and van Horssen (2015), and for $\delta=0$ and m=1, we get similar results as obtained in a study by Suweken and van Horssen (2003).

3.2. Detuning case: $\Omega = m\pi + \varepsilon\sigma$

In this section, we will investigate the (in) stability of the damped string-like equation in the neighborhood of resonance that is $\Omega \sim m\pi + \varepsilon\sigma$, where $m = 1,3,5 \cdots$. Thus we can write,

$$\Omega = m\pi + \varepsilon\sigma \tag{34}$$

where σ is a detuning parameter and taken to be of $O(1)$ and ε $(0 < \varepsilon \ll 1)$ is a small dimensionless parameter. By putting Eq. 34 into Eq. 23, the $O(\varepsilon)$-problem for v_{k1} is obtained:

$$\frac{\partial^2 v_{k1}}{\partial t_0^2} + (k\pi)^2 v_{k1} = -2\frac{\partial^2 v_{k0}}{\partial t_0 \partial t_1} - \delta\frac{\partial v_{k0}}{\partial t_0} + \sum_{\substack{n=1 \\ n\pm k \text{ is odd}}}^{\infty}\left(\frac{nk}{n^2-k^2}\right)\left[4\alpha(m\pi + \varepsilon\sigma)\cos((m\pi + \varepsilon\sigma)t_0)v_{n0} + 8(V_0 + \alpha\sin(m\pi + \varepsilon\sigma)t_0)\frac{\partial v_{n0}}{\partial t_0}\right]. \tag{35}$$

To prevent the unbounded terms, A_{k0} and B_{k0} have to satisfy,

$$\frac{dA_{k0}}{dt_1} = -\frac{\bar{\delta}}{2}A_{k0}(t_1) + \left[(m+k)\left(A_{(m+k)0}\sin\sigma t_1 + B_{(m+k)0}\cos\sigma t_1\right) - (m-k)\left(A_{(m-k)0}\sin\sigma t_1 + B_{(m-k)0}\cos\sigma t_1\right) - (k-m)\left(A_{(k-m)0}\sin\sigma t_1 - B_{(k-m)0}\cos\sigma t_1\right)\right]$$

$$\frac{dB_{k0}}{dt_1} = -\frac{\bar{\delta}}{2}B_{k0}(t_1) - \left[(m+k)\left(A_{(m+k)0}\cos\sigma t_1 - B_{(m+k)0}\sin\sigma t_1\right) + (m-\right.$$

$k)\big(A_{(m-k)0}\cos\sigma t_1 - B_{(m-k)0}\sin\sigma t_1\big) + (k - m)\big(A_{(k-m)0}\cos\sigma t_1 + B_{(k-m)0}\sin\sigma t_1\big)\big]$

where $\bar{t}_1 = \frac{\alpha t_1}{m}, \bar{\delta} = \frac{\delta m}{\alpha}$ and $k = 1,2,3,\dots$ and the functions A_{k0} and B_{k0} are considered to be zero for non-positive integers. For simplicity, the bar from $\bar{t}_1$ and $\bar{\delta}$ is dropped. It can be observed that for detuning parameter $\sigma = 0$ in Eq. 36, we get the same system given in Eq. 25. If damping coefficient $\delta = 0$ and detuning parameter $\sigma = 0$ Eq. 36 reduces to same equation obtained in the study by Malookani and van Horssen (2015).

3.2.1. Mathematical analysis of coupled system (Eq. 36)

In this subsection, we compute the energy from the coupled system of ODEs. (Eq. 36), which is obtained for avoiding the secular terms for the detuning case. Using transformations $X_{k0}(t_1) = kA_{k0}$ and $Y_{k0}(t_1) = kB_{k0}$ with $k = 1,2,3\cdots$, thus the coupled system of ODEs (Eq. 36) yields:

$\frac{dX_{k0}}{dt_1} = -\frac{\bar{\delta}}{2}X_{k0} + k\big[(X_{(m+k)0}\sin\sigma t_1 + Y_{(m+k)0}\cos\sigma t_1) - (X_{(m-k)0}\sin\sigma t_1 + Y_{(m-k)0}\cos\sigma t_1) - (X_{(k-m)0}\sin\sigma t_1 - Y_{(k-m)0}\cos\sigma t_1)\big]$ (37)

$\frac{dY_{k0}}{dt_1} = -\frac{\delta}{2}Y_{k0} + k\big[(-X_{(m+k)0}\cos\sigma t_1 + Y_{(m+k)0}\sin\sigma t_1) - (X_{(m-k)0}\cos\sigma t_1 - Y_{(m-k)0}\sin\sigma t_1) - (X_{(k-m)0}\cos\sigma t_1 + Y_{(k-m)0}\sin\sigma t_1)\big].$

The functions $X_{k0}(t_1)$ and $Y_{k0}(t_1)$ are zero for non-positive integers. Then from Eq. 37, it is concluded that,

$X_{k0}\dot{X}_{k0} = -\frac{\delta}{2}X_{k0}^2 + k\big[X_{k0}(X_{(m+k)0}\sin\sigma t_1 + Y_{(m+k)0}\cos\sigma t_1) - X_{k0}(X_{(m-k)0}\sin\sigma t_1 + Y_{(m-k)0}\cos\sigma t_1) - X_{k0}(X_{(k-m)0}\sin\sigma t_1 - Y_{(k-m)0}\cos\sigma t_1)\big]$ (38)

$Y_{k0}\dot{Y}_{k0}$
$= -\frac{\delta}{2}Y_{k0}^2$
$+ k\big[Y_{k0}(-X_{(m+k)0}\cos\sigma t_1 + Y_{(m+k)0}\sin\sigma t_1)$
$- Y_{k0}(X_{(m-k)0}\cos\sigma t_1 - Y_{(m-k)0}\sin\sigma t_1)$
$- Y_{k0}(X_{(k-m)0}\cos\sigma t_1 + Y_{(k-m)0}\sin\sigma t_1)\big].$

By taking sum on both sides of equations in Eq. 38, and then by the sum up from $k = 1$ to ∞, it yields,

$\frac{1}{2}\frac{d}{dt_1}\sum_{k=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) + \frac{\delta}{2}\sum_{K=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) = m\sum_{K=1}^{\infty}\big[(X_{(k+m)0}Y_{k0} - X_{k0}Y_{(k+m)0})\cos\sigma t_1 - (X_{k0}X_{(k+m)0} + Y_{k0}Y_{(k+m)0})\sin\sigma t_1\big] + (1)\big[(Y_{(m-1)0}Y_{10} - X_{10}X_{(m-1)0})\sin\sigma t_1 - (X_{10}Y_{(m-1)0} + X_{(m-1)0}Y_{10})\cos\sigma t_1\big] + (2)\big[(Y_{(m-2)0}Y_{20} - X_{20}X_{(m-2)0})\sin\sigma t_1 - (X_{20}Y_{(m-2)0} + X_{(m-2)0}Y_{20})\cos\sigma t_1\big] + \cdots + (m-2)\big[(Y_{(m-2)0}Y_{20} - X_{20}X_{(m-2)0})\sin\sigma t_1 - (X_{20}Y_{(m-2)0} + X_{20}Y_{(m-2)0})\cos\sigma t_1\big] + (m-1)\big[(Y_{(m-1)0}Y_{10} - X_{10}X_{(m-1)0})\sin\sigma t_1 - (X_{10}Y_{(m-1)0} + X_{10}Y_{(m-1)0})\cos\sigma t_1\big].$ (39)

Differentiate the Eq. 39 w.r.t t_1 two times, we get,

$\frac{d^3w}{dt_1^3} + 3\delta\frac{d^2w}{dt_1^2} + (3\delta^2 - 4m^2 + \sigma^2)\frac{dw}{dt_1} + (\delta^3 - 4m^2\delta + \sigma^2\delta)w = 0$ (40)

where $\sum_{K=1}^{\infty}(X_{k0}^2 + Y_{k0}^2) = w(t_1)$ and as $-\delta$ is the root of Eq. 40 thus it yields,

$\frac{d^2w}{dt_1^2} + 2\delta\frac{dw}{dt_1} + (\delta^2 - 4m^2 + \sigma^2)w = 0.$ (41)

Finally, the roots of Eq. 41 are $-\delta \pm \sqrt{4m^2 - \sigma^2}$ so we have different cases for damping and detuning parameter as under:

Case 1: when $4m^2 - \sigma^2 = 0$ that is, $|\sigma| = 2m$, then $w(t_1) = (C_1 + C_2t_1 + C_3t_1^2)e^{-\delta t_1}$ where C_1, C_2, C_3 are arbitrary constants. Further, if $\delta > 0$, the energy of the system tends to zero for sufficiently large time, and the system remains stable, however, for $\delta = 0$ the energy grows polynomially, and the system becomes unbounded.

Case 2: when $4m^2 - \sigma^2 > 0$ that is $|\sigma| < 2m$ then $w(t_1) = C_1e^{-\delta t_1} + e^{-\delta t_1}(C_2\cosh\sqrt{4m^2 - \sigma^2}t_1 + C_3\sinh\sqrt{4m^2 - \sigma^2}t_1)$ and for $\delta > \sqrt{4m^2 - \sigma^2}$ the energy of system reduces and system remains stable, however for $\delta < \sqrt{4m^2 - \sigma^2}$ the energy of system grows exponentially, thus system becomes unbounded.

Case 3: when $4m^2 - \sigma^2 < 0$ that is $|\sigma| > 2m$ then $w(t_1) = C_1e^{-\delta t_1} + e^{-\delta t_1}(C_2\cos\sqrt{4m^2 - \sigma^2}t_1 + C_3\sin\sqrt{4m^2 - \sigma^2}t_1)$ thus, the energy of the system is always bounded due to the trigonometric solutions.

4. Results and discussion

In this paper, (in) stability of the vertical vibrations of the string-like model under the effect of viscous damping have been studied. The string is moving in only one positive x-direction with $V(t) = \varepsilon(V_0 + \alpha\sin(\Omega t))$, where $0 < \varepsilon \ll 1$ and V_0, α, Ω are positive constants. In order to find approximate-analytic solutions of the governing equations of motion, a Fourier expansion method, together with two timescales perturbation method, is employed. It is found out that there are infinitely many values of parameter Ω, giving rise to resonances. The general resonance case that is, $\Omega = m\pi$, and the (Near resonance) detuning case that is, $\Omega = m\pi + \varepsilon\sigma$, where σ is detuning parameter and $m = 1,3,5,\dots$ has been studied in detail.

The energy of an infinite dimensional system is obtained from a coupled system of ODEs, and it is observed that for $\delta = 2m$ energy of system decreases and eventually become constant, for $\delta > 2m$ the energy of system tends to zero as times increases and for $\delta < 2m$ energy grows without bound and finally for $\delta = 0$ similar solution is obtained by Malookani and van Horssen (2015). Eigenvalues of a coupled system for $m = 3$ and $m =$

5 are obtained up to 10 modes by using computer software Maple16, which are in a multiplicity of 2. For $\delta \geq 2m$, 1st, 4th, 7th, 8th, 9th, and 10th mode-amplitudes are stable due to the negative real part as given in Table 1.

However, 2nd, 3rd, mode-amplitudes are stable for damping parameter $\delta > 2.828$ and 5th, and 6th modes are stable for damping parameters $\delta > 0.80$ and similarly for $\delta \geq 2m$, 1st, 2nd, and 7th are mode-amplitudes are stable in nature. However, 3rd, 4th, 5th, 6th, are stable for damping parameter $\delta > 4.89$ and 8th, 9th, 10th, modes are stable for damping parameter $\delta > 2.16$ as shown in Table 2. For $\delta < 2m$, the energy of the system and mode-truncation behaves differently, so mode-truncation for this case is not possible.

In addition, the detuning case for $m > 1$ has also been discussed in detail. For $|\sigma| = 2m$ and $\delta > 0$ the energy of system decays as time increases and system remains stable. If $\delta = 0$, the energy grows polynomially, and systems remains unbounded. For $|\sigma| < 2m$ and $\delta > \sqrt{4m^2 - \sigma^2}$, energy of the system decays and system is stable but for $\delta < \sqrt{4m^2 - \sigma^2}$ energy of coupled system grows exponentially, so the system is unstable for this case. Finally, for $|\sigma| > 2m$, the energy is always bounded due to the trigonometric terms.

5. Concluding remarks

The stability analysis is carried out in the vertical vibrations of the string-like model under the influence of viscous damping, with time-varying velocity. The Fourier expansion method, together with two timescales perturbation method, has been used for the search of approximate-analytic solutions. It has been found that there are infinitely many values of fluctuation parameter Ω, which generates the resonance in the system.

This study has focused on the stability of the string-like model at the general resonance and the resonant detuning cases. It has been found that the energy obtained from a coupled system of ODEs decays only for two cases of damping parameter δ, that is, $\delta = 2m$ and $\delta > 2m$. The eigenvalues of a coupled system computed up to 10 modes, which are stable depending on the certain values of the damping parameter. It turned out that the Galerkin's truncation is not problematic for these two cases of damping, whereas for $\delta < 2m$ Galerkin's truncation is not applicable due to different behavior of energy and modes.

Finally, for the detuning case, it has been observed that for detuning parameter σ, that is, for $|\sigma| > 2m$ the energy of the system is bounded, whereas, for $|\sigma| < 2m$ with $\delta > \sqrt{4m^2 - \sigma^2}$ the energy tends to zero as the time progresses and for $\delta < \sqrt{4m^2 - \sigma^2}$, the energy grows exponentially. For $|\sigma| = 2m$ with $\delta > 0$, the energy of the system damps out, whereas, for $\delta = 0$ the energy grows polynomial.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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