Nonlinear integral equations solution method based on operational matrices of Chebyshev

Jumah Aswad Zarnan *

Department of Accounting by IT, Cihan University, Sulaimaniya, Kurdistan, Iraq

ABSTRACT

In this paper, the solution of Hammerstein integral equations is presented by a new approximation method based on operational matrices of Chebyshev polynomials. The nonlinear Hammerstein and Volterra Hammerstein integral equations are reduced to a system of nonlinear algebraic equations by using operational matrices of Chebyshev polynomials. Illustrative examples are presented to test the method. The method is less complicated in comparison to others. The results obtained are demonstrated with previously validated results.

1. Introduction

In this work, we consider the nonlinear integral equations of Hammerstein and Volterra-Hammerstein types.

The nonlinear integral equations of Hammerstein and Volterra-Hammerstein types are considered, which they have the following forms respectively:

$$u(x) = f(x) + \lambda \int_0^1 k(x,s)\psi(s,u(s))ds \qquad (1)$$
$$u(x) = f(x) + \lambda \int_0^x k(x,s)\psi(s,u(s))ds. \qquad (2)$$

Assume that f, ψ and k are to be in L^2 and λ is a real known constant, and $\psi(x, u(s))$ nonlinear in u. We assume that Eqs. 1 and 2 have a unique solution u(x) to be determined.

Several problems in mathematical physics, boundary value problems, and in the theory of elasticity are reduced into Volterra Hammerstein integral equations (Eq. 2) (Abdou, 2003; Ganesh and Joshi, 1991). The nonlinear Hammerstein integral equations (Eq. 1) arise as a reformulation of two-point boundary value problems with a certain nonlinear boundary condition, (Delves and Mohamed 1985).

The aim of the present paper is to consider the numerical solution of the nonlinear integral equations of Hammerstein and Volterra-Hammerstein types based on the Chebyshev approximation (Zarnan, 2016).

Several numerical methods for approximating the solution of Hammerstein integral equations were considered as a review for this work. The classical method of successive approximations for the solution of Fredholm-Hammerstein's integral equations was introduced in Tricomi (1985). Lardy (1981), and Kumar & Sloan (1987) introduced A variation of the Nystrom method and collocation-type method. The discussion of the connection of nonlinear Volterra-Hammerstein integral equations and integro-differential equations with iterated collocation method and application of a collocation-type method is presented by Brunner (1992). The asymptotic error expansion of a collocation-type method for Volterra-Hammerstein's integral equations is discussed by Han (1993). The Walsh series operational matrix of integration to solve linear integral equations was described by Hsiao and Chen (1979).

Moreover, Hwang and Shih (1982), Chang and Wang (1985), Chou and Horng (1985) and Razzaghi et al. (1990) used the operational matrices of integration associated with Laguerre polynomials, Legendre polynomials, Chebyshev polynomials, and Fourier series to derive continuous solutions for linear integral equations in Babolian et al. (2007) a computational method for solving nonlinear Fredholm–Volterra Hammerstein integral equations is described.

The collocation-type method and rationalized Haar function to nonlinear Volterra–Fredholm–Hammerstein integral equations are introduced

(Brunner, 1992; Ordokhani, 2006). Yousefi and Razzaghi (2005) applied Legendre wavelets to a special type of nonlinear Volterra–Fredholm integral equations. Han (1993) introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra–Hammerstein integral equations.

Orthogonal functions often used to represent arbitrary time functions, have received considerable in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving systems of algebraic equations, thus greatly simplifying the problem. Orthogonal functions have also been proposed to solve linear integral equations. Special attentions have been given to applications of Walsh functions (Hsiao and Chen, 1979), Block-pulse functions (Wang and Shih, 1982), Laguerre series (Wang and Shih, 1982), and Chebyshev polynomials (Chou and Horng, 1985). Very few references have been found in technical literature dealing with Volterra–Fredholm integral equations. Yalçinbaş (2002) applied a Taylor series to the nonlinear Volterra–Fredholm integral equation.

2. Elements of a research

2.1. Chebyshev polynomials

A sequence of orthogonal polynomials that are related to de Moivre's formula and which can be defined recursively is called the Chebyshev polynomials. In Saran et al. (2000) the nth degree of the Chebyshev polynomials defined by:

$$T_n(x) = \sum_{m=0}^{[n/2]}(-1)^m \frac{n!}{(2m)!(n-2m)!}(1-x^2)^m x^{n-2m}, \quad (3)$$

where, $[n/2] = \begin{cases} n/2 & \text{if } n \text{ is evevn} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$ and,

$$T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1, T_5(x)$$
$$= 16x^5 - 20x^3 + 5x, \quad T_6(x)$$
$$= 32x^6 - 48x^4 + 18x^2 - 1.$$

In Fig. 1, we give the first six Chebyshev polynomials over the interval [-1, 1].
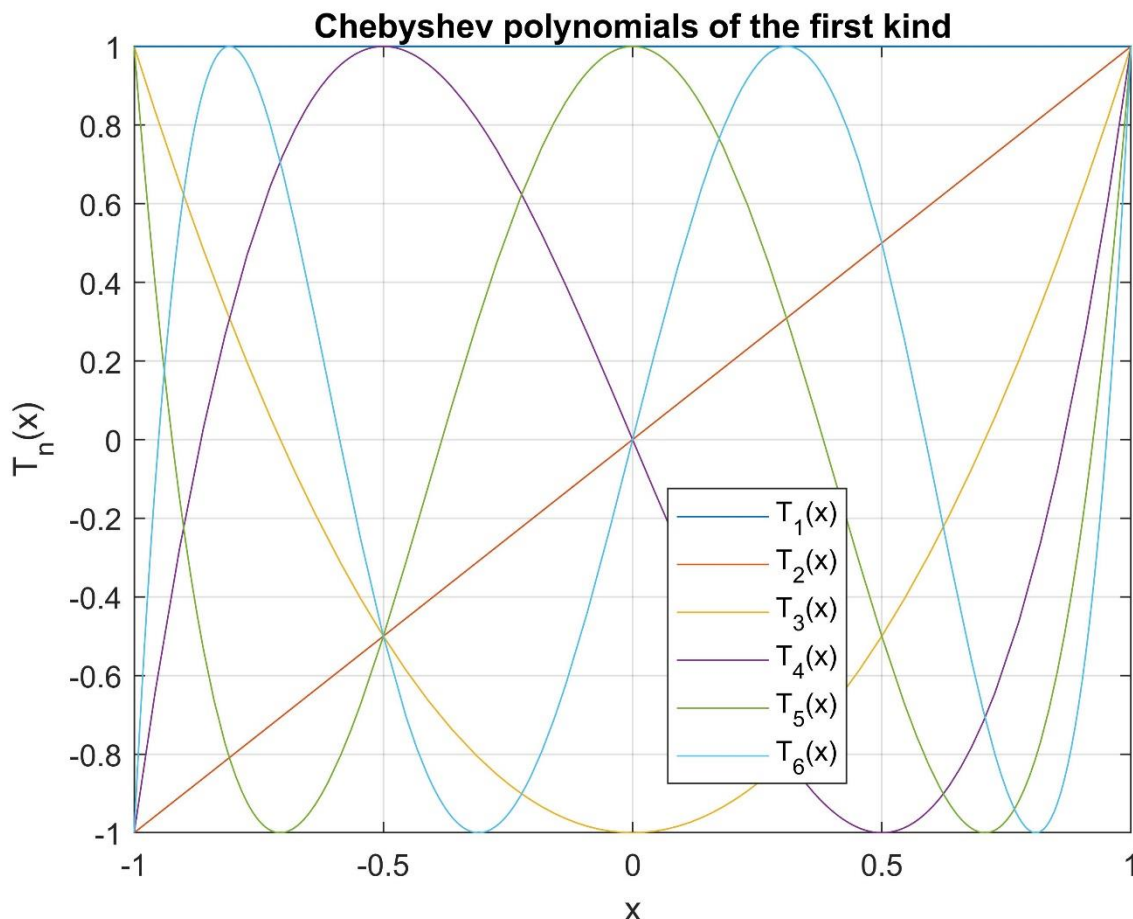

Fig. 1: First 6 Chebyshev polynomials over the interval [-1, 1]

Now, we define,

$$\Phi(x) = [T_{0,n}(x), T_{1,n}(x), \dots, T_{n,n}(x)]^T, \quad (4)$$

where,

$$\Phi(x) = AB_n(x), \quad (5)$$

where, A is an $(n + 1) \times (n + 1)$ upper triangular matrix with rows,

$$\left[\overbrace{0,0,\dots 0}^{i\ times}, (-1)^0 \binom{n}{i}\binom{n-i}{0}, (-1)^1 \binom{n}{i}\binom{n-i}{1}, \dots, (-1)^{m-i}\binom{n}{i}\binom{n-i}{n-i} \right],$$

and $B_n(x)$ is an $(n + 1) \times 1$ matrix as follows:

$$B_n(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}.$$

2.2. Function approximation

Obtaining approximate function values much quicker with an approximating functional form. Assume that the Hilbert space with the inner product is $H = L^2[0,1]$ and defined by:

$$(f, g) = \int_0^1 f(x)g(x)dx,$$

and,

$$Y = Span\{T_{0,n}(x), T_{1,n}(x), \dots, \quad T_{n,n}(x)\},$$

where, Y is a complete subspace of H if it a finite-dimensional and closed subspace.

Now f has a unique best approximation out of Y such as y_0, if it is an arbitrary element in H, that is (Yousefi and Behroozifar, 2010):

$$\exists y_0 \in Y \quad s.t \quad \forall y \in Y \quad \|f - y\|,$$

this gives,

$$\forall y \in Y \quad (f - y_0) = 0. \tag{6}$$

Since $\forall y \in Y$ so there exist coefficients $c_0, c_1, \dots, c_n$ such that:

$$y_0 = c^T \Phi(x),$$

where,

$$c^T = [c_0, c_1, \dots, c_n]. \tag{7}$$

Using Eq. 6:

$$\left(f - c^T\Phi(x), T_{i,n}(x)\right) = 0, \quad i = 0, 1, \dots, n,$$

if we write,

$$\left(c^T(\Phi(x), \Phi(x)\right) = \left(f, \Phi(x)\right), \tag{8}$$

where,

$$\left(f, \Phi(x)\right) = \int_0^1 f(x)\Phi(x)dx.$$

Here, the dual matrix of $\Phi(x)$ is the $(n + 1) \times (n + 1)$ $(\Phi(x), \Phi(x))$. Assume that:

$$D = \left(\Phi(x), \Phi(x)\right)A\left[\int_0^1 T_n(x)T_n^T(x)dx\right]A^T = AHA^T, \tag{9}$$

H is a Hilbert matrix. We can write the element of D as:

$$D_{(i+1),\ (j+1)} = \int_0^1 T_{i,n}(x)T_{j,n}(x)dx = \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}}, \tag{10}$$

where $i, j = 0, 1, \dots, n$. Any function $f(x) \in L^2[0,1]$ can be written using Chebyshev basis as $f(x) \simeq c^T(\Phi(x),$ where from (8) and (9), we receive

$$c = D^{-1}\left(f, \Phi(x)\right). \tag{11}$$

The function $k(x, s) \in L^2[0,1] \times L^2[0,1]$ can be approximate as:

$$k(x, s) \simeq \Phi^T(x)K\Phi(s), \tag{12}$$

where,

$$K_{i,j} = \frac{\left(\Phi_i(x), \left(k(x,s), \Phi_j(s)\right)\right)}{\left(\Phi_i(x), \Phi_i(x)\right)\left(\Phi_j(x), \Phi_j(x)\right)}, \tag{13}$$

is an $(n + 1) \times (n + 1)$ matrix, For $i, j = 0, 1, \dots, n$. From Eq. 9:

$$K = D^{-1}\left(\Phi_i(x), \left(k(x,s), \Phi(s)\right)\right)D^{-1}. \tag{14}$$

2.3. Operational matrix of integration

The approach is based on reducing the differential equations into integral equations through integration, approximating various signals involved in the equation by truncated orthogonal series and using the operational matrix of integration, to eliminate the integral operations.

Integrated the vector $\Phi(x)$ in Eq. 4 we obtain:

$$\int_0^x \Phi(x')d\,x' \simeq P\,\Phi(x). \tag{15}$$

The $(n + 1) \times (n + 1)$ operational matrix for integration P in Eq. 15 is given in (Wazwaz, 2011) as:

$$\int_0^x \Phi(x')d\,x' = A_p X_p, \tag{16}$$

and A_p is the $(n + 1) \times (n + 1)$ matrix,

$$A_p = A \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{n+1} \end{bmatrix} and\ X_p = \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^{n+1} \end{bmatrix}. \tag{17}$$

The elements of the vector X_p in terms of $\Phi(x)$ can be approximated as: From Eq. 5, $T_n(x) = A^{-1}\Phi(x)$, then for $k = 0, 1, \dots, n$, $x^k = A_{[k+1]}^{-1}\Phi(x)$, where, $A_{[k+1]}^{-1}$ is $k + 1 -$ th row for A^{-1} for $k = 0, 1, \dots, n$. We just need to approximate $x^{n+1} \simeq c_{n+1}^T\Phi(x)$. By using (11) and (10), we have:

$$c_{n+1} = D^{-1} \int_0^1 x^{n+1} \Phi(x) dx = \frac{D^{-1}}{2n+2} \begin{bmatrix} \frac{\binom{n}{0}}{\binom{2n+1}{n+1}} \\ \frac{\binom{n}{1}}{\binom{2n+1}{n+2}} \\ \vdots \\ \frac{\binom{n}{n}}{\binom{2n+1}{2n+1}} \end{bmatrix}$$

let,

$$T = \begin{bmatrix} A^{-1}_{[2]} \\ A^{-1}_{[3]} \\ \vdots \\ A^{-1}_{[n+1]} \\ c^T_{n+1} \end{bmatrix}, \tag{18}$$

then $X_p \simeq T\Phi(x)$. Therefore, the operational matrix of integration is given by:

$$P = A_p T.$$

2.4. Product operational matrix

It is important to calculate the product of $\Phi(x)$ and $\Phi(x)^T$, named the product matrix of Chebyshev polynomials basis. Assume:

$$\Pi(x) = \Phi(x)\Phi(x)^T. \tag{19}$$

If the matrix $\Pi(x)$ is multiplying in vector c defined in Eq. 7, we have:

$$c^T \Pi(x) = \Phi(x)^T \hat{C}, \tag{20}$$

where, $\hat{C}$ is an $(n + 1) \times (n + 1)$ matrix and called the coefficient matrix. So we have:

$$c^T \Pi(x) = \\ \left[\sum_{i=0}^n c_i T_{i,n}(x), \sum_{i=0}^n c_i x T_{i,n}(x), \dots, \sum_{i=0}^n c_i x^n T_{i,n}(x), \right] A^T. \tag{21}$$

To approximate the function $x^k T_{i,n}(x)$ in terms of $\Phi(x)$, we assume that:

$$e_{k,i} = \begin{bmatrix} e_0^{k,i} \\ e_1^{k,i} \\ \vdots \\ e_n^{k,i} \end{bmatrix}. \tag{22}$$

Using Eq. 11, we have $x^k T_{i,n}(x) \simeq e_{k,i}\Phi(x)$, $i, k = 0, 1, \dots, n$. By using Eqs. 11 and 10 for $i, k = 0, 1, \dots, n$, we have:

$$e_{k,i} = D^{-1} \int_0^1 x^k T_{i,n}(x)\Phi(x) dx = \frac{D^{-1}\binom{n}{i}}{2n+k+1} \begin{bmatrix} \frac{\binom{n}{0}}{\binom{2n+k}{i+k}} \\ \frac{\binom{n}{1}}{\binom{2n+k}{i+k+1}} \\ \vdots \\ \frac{\binom{n}{n}}{\binom{2n+k}{i+k+n}} \end{bmatrix}, \tag{23}$$

therefore,

$$\sum_{i=0}^n c_i x^k T_{i,n}(x) \simeq \sum_{i=0}^n c_i \left(\sum_{j=0}^n e_j^{k,i} T_{i,n}(x) \right) = \\ \Phi(x)^T \begin{bmatrix} \sum_{i=0}^n c_i e_0^{k,i} \\ \sum_{i=0}^n c_i e_1^{k,i} \\ \vdots \\ \sum_{i=0}^n c_i e_n^{k,i} \end{bmatrix} \tag{24}$$

$$= \Phi(x)^T [e_{k,0}, e_{k,1}, \dots, e_{k,n}] c = \Phi(x)^T E_{k+1} c,$$

where, $E_{k+1}c$ is an $(n + 1) \times (n + 1)$ matrix, that has vectors $e_{k,i}$, $k = 0,1, \dots, n$ for each column's. Then we define $\widetilde{E_{k+1}} = E_{k+1}c$ for k=0,1,…,n. If we choose an $(n + 1) \times (n + 1)$ matrix $\tilde{C} = [\widetilde{E_1}, \widetilde{E_2}, \dots, \widetilde{E_{n+1}}]$, then by Eqs. 21 and 24 we have:

$$c^T \Pi(x) \simeq \Phi(x)^T \tilde{C} A^T. \tag{25}$$

3. Solution of Hammerstein integral equations

Integral equations of the Fredholm–Hammerstein are the most important applications of the methods of nonlinear functional analysis and of the theory of nonlinear operators. This kind of integral equation appears in nonlinear physical such as electro-magnetic fluid dynamics, reformulation of boundary value problems (BVPs) with a nonlinear boundary condition.

For solving Hammerstein Integral Eq. 1, we let:

$$z(s) = \psi(s, u(x)), \quad 0 \le s \le 1, \tag{26}$$

then we get,

$$u(x) = f(x) + \lambda \int_0^1 k(x,s)z(s)ds, \tag{27}$$

substituting Eq. 27 in Eq. 26, we get:

$$z(x) = \psi(x, f(x) + \int_0^1 k(x,s)z(s)ds. \tag{28}$$

We approximate this equation as:

$$z(x) = Z^T \Phi(x), \tag{29}$$

where, and $\Phi(x)$ are defined with Eqs. 4 and 7. Using Eqs. 9, 12, and 29, we have:

$$\int_0^1 k(x,s)z(s)ds \simeq \int_0^1 \Phi^T(x)K\Phi(s)\Phi(s)^T Z ds$$
$$= \Phi^T(x)K \int_0^1 \Phi^T(x)\Phi(s)^T Z ds)Z$$
$$= \Phi^T(x)KDZ. \tag{30}$$

Via Eqs. 28, 29, and 30, we get,

$$Z^T \Phi(x) = \psi(x, f(x) + \lambda \Phi^T(x)KDZ). \tag{31}$$

In order to find Z we collocate Eq. 31 in n nodal points of Newton-cotes as:

$$x_p = \frac{2p-1}{2n}, \quad p = 1, 2, \dots, n. \tag{32}$$

Then, we have Eq. 31 as follows:

$$Z^T \Phi(x_p) = \psi(x_p, f(x_p) + \lambda \Phi^T(x_p)KDZ). \tag{33}$$

We can calculate the unknown vector Z from the above nonlinear system of equations. Newton's iterative method is suitable for solving this nonlinear system. We used Mathematica 7 to obtain our solutions. The required approximated solution $u(x)$ for our Hammerstein integral equation (Eq. 1), can be obtained by using Eqs. 27 and 29 as follows:

$$u(x) = f(x) + \lambda \Phi^T(x)KDZ). \tag{34}$$

4. Solution of nonlinear Volterra Integral equations

Linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics spread of epidemics, and semi-conductor devices. Volterra started working on integral equations.

Consider the nonlinear Volterra integral equations given in Eq. 2. For solving these equations, we let $z(s) = \psi(s, u(s))$ for $0 \le s \ge 1$. Then from Eq. 2, we get:

$$z(x) = \Psi(x, f(x) + \int_0^x k(x,s)z(s)ds). \tag{35}$$

Using Eqs. 20, 12, 29, and 15 we can write,

$$
\begin{aligned}
\int_0^x k(x,s)z(s)ds &\simeq \int_0^x \Phi^T(x)K\Phi(s)\Phi(s)^T Zds \\
&= \Phi^T(x)K \int_0^x \Pi(s)Zds \\
&= \Phi^T(x)K\hat{Z}^T \int_0^x \Phi(s)ds \\
&= \Phi^T(x)K\hat{Z}^T P\Phi(s).
\end{aligned} \tag{36}
$$

After using Eq. 36, we get:

$$Z^T \Phi(x) = \psi(x, f(x) + \lambda \Phi^T(x)KDZ)\hat{Z}^T P\Phi(s)). \tag{37}$$

Collocating Eq. 37 in n nodal point (Eq. 32), we have,

$$Z^T \Phi(x_p) = \psi(x_p, f(x_p) + \lambda \Phi^T(x_p)K\hat{Z}^T P\Phi(x_p)) \tag{38}$$
for $p = 1, 2, \ldots, n.$

After solving the nonlinear system (Eq. 38), we get Z, and by use of $z(x) = Z^T \Phi(x_p)$ we will have the approximation solution of Volterra–Hammerstein integral equations (Eq. 2) as:

$$u(x) = f(x) + \int_0^x k(x,s)z(s)ds. \tag{39}$$

From above, we know that Eq. 39 can be evaluated by

$$u(x) = f(x) + \lambda \Phi^T(x)K\hat{Z}^T P\Phi(x). \tag{40}$$

The main aim in the present approach is to reduce such problems to those of solving systems of algebraic equations with constant coefficients by unknown variables. The validity and effectiveness of the approach are demonstrated via some numerical examples

5. Numerical examples

To illustrate the effectiveness of the presented approach in the present paper, some test examples are carried out in this section. The results obtained by the present methods reveal that the present method is very effective and convenient for nonlinear Fredholm integral equations.

5.1. Example 1

Consider the nonlinear Fredholm integral equation:

$$u(x) = f(x) + \int_0^1 2x^2 sin(u(S))ds,$$

where,

$$f(x) = 1 + x + \left(1 - \frac{3}{2}ln(3) + \frac{\sqrt{3}}{6}\pi\right)x^2.$$

The exact solution is $1 + x + x^2$. Table 1 shows the present method results, for example, 1 in comparison with the method of Mahmoudi (2005). The superiority of the Chebyshev operational matrices method compared with the Taylor polynomial method is clear here because, with the same number of basic functions, we get very better results.

Table 1: Approximate and exact solutions for example 1

x_i	Present method n=6	Method of Kumar and Sloan (1987) N = 6	Exact solution
0.0	1.000000	1.000000	1
0.2	1.239987	1.238432	1.24
0.4	1.559988	1.553726	1.56
0.6	1.959988	1.945884	1.96
0.8	2.439989	2.414905	2.44
1.0	2.999999	2.960788	3

In this table, the exact and the computed solutions have been given. As seen, the computed solution is in good agreement with the exact solution. It is clear that the presented approach can be considered as an efficient approach to solving the nonlinear Fredholm integral equations.

5.2. Example 2

Consider the Hammerstein integral equation:

$$u(x) = f(x) + \int_0^1 sin(x+s)\,ln(u(s))\,ds, \quad 0 \le x \le 1,$$

where, $f(x) = e^x - 0.382 sin(x) - 0.301 cos(x)$, and the exact solution is $u(x) = e^x$ (Brunner, 1992). The computational results are obtained by the present method with $n = 5$, and we compared our results by the results of Brunner (1992). In this comparison, the number of present method basis functions is 5,

but the number of basic functions for method of Maleknejad et al. (2010a) is 32, and the results have almost same accuracy, so Chebyshev method is superior to hybrid Legendre and Block–Pulse method for solving Hammerstein integral equation. Table 2 shows approximate and exact solutions, for example 2.

Table 2: Approximate and exact solutions for example 2

x_i	Present method n=5	Method of Brunner (1992) m=4, n=8	Exact solution
0.0	1.0001824211	1.0001817942	1
0.2	1.2215473602	1.2215472834	1.2214027582
0.4	1.4919260323	1.4919261952	1.4918246976
0.6	1.8221730237	1.8221731864	1.8221188004
0.8	2.2255460320	2.2255459923	2.2255409285
1.0	2.7182380265	2.7182373557	2.7182818285

In this table, the exact and the calculated solutions have been given. As seen, the calculated solution is in good agreement with the exact solution. It is clear that the presented method can be considered as an efficient method to solve the Hammerstein integral equations.

5.3. Example 3

Table 3 shows errors E2 for example 3. Consider the nonlinear Volterra integral equation:

$$u(x) = \frac{3}{2} - \frac{1}{2}e^{-2x} + \int_0^x \big(u^2(s) + u(s)\big)ds,$$

where, the exact solution is e^{-x}.

The L^2-norm of errors is considered in this example, which can be given by:

$$E_2 = \left(\int_0^1 [u(x) - u_n(x)]^2 dx\right)^{1/2}.$$

Table 3: Errors E2 for example 3

n|m	Present method	Method of Han (1993)
4	0.000068193251	0.003738014268
8	0.000000084171	0.000937018240
16	0.000000000000	0.000937018240
32	0.000000000000	0.002374324588

In Table 3, the comparison among the Chebyshev operational matrices errors E2 with n=4, 8, 16, 32 beside triangular function (Maleknejad et al., 2010b) errors with m=4, 8, 16, 32 are given. The primacy of the present method compared with the triangular function method is obvious here because by the same number of basis function present method E2 errors are very low.

6. Conclusion

This work presents a numerical solution approach for Hammerstein and Volterra Hammerstein integral equations by the operational matrices of Chebyshev polynomials. It had been proven in this research that a nonlinear system of algebraic equations is possible to be solved by Newton's method. The less complicity and attractiveness of the Chebyshev polynomials operational matrices method are clear in the present paper. The implementation of the current approach in analogy to existed methods is more convenient, and the accuracy is higher than previous methods. Numerical examples are given to demonstrate the validity and applicability of the proposed method.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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