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Dynamical properties of a 2-D non-invertible system

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ABSTRACT

The non-invertible systems are very useful in practical applications. The study of the non-invertible systems has important value since a large number of genetics studies in biology, physics, engineering, and economic systems have been widely carried out found to exhibit a class of non-invertible systems. This short paper proposes a new simple four-term 2-D polynomial chaotic system with only one quadratic nonlinearity and describes its some interesting dynamical properties. Moreover, the stability of the fixed point and chaotic motions are investigated using analytical and numerical methods. Our 2-D polynomial system displays new chaotic attractors via the quasi-periodic route to chaos for certain values of its parameter of bifurcation.

1. Introduction

Many papers have described 2-D chaotic invertible system with a quadratic inverse and constant Jacobian (Aziz-Alaoui et al., 2001; Miller and Grassi, 2001), one of the most famous is the smooth two-dimensional Hénon system (Hénon, 1976) and studied in detail by others (Hénon, 1969; Benedicks and Carleson, 1991; Cao and Liu, 1998; Marotto, 1979). In this context, the study of the non-invertible systems has important value since, a large number of genetics researches in biology (Bi and Ruan, 2013; Gałach, 2003), physics (Benerjee and Verghese, 2001), engineering (Tse, 2003), economics (Bischi and Tramontana, 2010), and applied mathematics (Mammeri, 2018) systems have been widely carried out found to exhibit a class of non-invertible quadratic systems. This short paper proposes a new simple 2-D non-invertible discrete chaotic system (2) with one bifurcation parameter, and that has only one nonlinear term (Mammeri, 2017). In section 1, a rigorous proof of the existence of some interesting properties of the system (2) on open, a connected subset is given, in section 2, a detailed dynamical behavior of this system (2) is further investigated numerically in term of a single bifurcation parameter. The final section concludes the letter.

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It is well known that the general form two-dimensional quadratic systems were made by Zeraoulia and Sprott (2010), where the 2-D quadratic systems are classified according to their number of nonlinearities. Also, many examples are given. And the first case of one nonlinearity is defined by:

$$\binom{x_{n+1}}{y_{n+1}} = \binom{a_0+a_1x_n+a_2y_n}{b_0+b_1x_n+b_2y_n+b_3x_ny_n} \quad (1)$$

In this paper, the new simplest two-dimensional quadratic system with only one cross-product nonlinear term xy is presented as follows:

$$f(x_n, y_n) = \binom{x_{n+1}}{y_{n+1}} = \binom{x_n - ay_n}{x_n - ax_ny_n}, \quad (2)$$

where $(x, y) \in \mathbb{R}^2$ and $a \in \mathbb{R}_+^*$ is the bifurcation parameter. For $a = 0$ the system (2) reduces to a two-dimensional linear system. On the other hand, the system (2) permits the construction of a new family of attractors dependent on the bifurcation parameter a and initial conditions.

2. Qualitative properties of the system

In the following section, we will prove some propositions in order to rigorously demonstrate the existence of some interesting properties of the system (2) on the largest open, connected subset. Let us define the following subset: $\Omega = \{(x, y) \in \mathbb{R}^2 : 1 - x - ay > 0\}$.

Proposition 1: The system (2) is invertible if

$$\binom{-a}{1} \neq \frac{1}{(n+1)\sum y_k^2 - (\sum y_k)^2} \binom{(n+1)\sum y_k x_k - \sum y_k \sum x_k}{-\sum y_k \sum y_k x_k + \sum y_k^2 \sum x_k}$$

Proof: The determinant of the Jacobi matrix of the system (2) evaluated at a point (x, y) is $detDf(x, y) = a(-ay + 1 - x)$ and we consider the finite number of points $(x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k), \ldots, (x_n, y_n)$ of an orbit of the system (2) and let us define the following matrix,

$$A = \begin{pmatrix} y_0 & 1 \\ y_1 & 1 \\ \vdots & \vdots \\ y_n & 1 \end{pmatrix}, Y = \binom{-a}{1}, B = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix},$$

then one has,

$$AY - B = \begin{pmatrix} -ay_0+1-x_0 \\ -ay_1+1-x_1 \\ \vdots \\ -ay_n + 1 - x_n \end{pmatrix},$$

we use the results available on linear algebra, then one has,

$$\|AY - B\|^2 = \sum_{k=0}^{n}(-ay_k + 1 - x_k)^2,$$

the system (2) is invertible if,

$$\|AY - B\|^2 \neq 0,$$

i.e.,

$$Y \neq ({}^tAA)^{-1}({}^tAB),$$

where,

$${}^tAA = \begin{pmatrix} y_0 \ldots y_n \\ 1 \ldots 1 \end{pmatrix}\begin{pmatrix} y_0 & 1 \\ y_1 & 1 \\ \vdots & \vdots \\ y_n & 1 \end{pmatrix} = \begin{pmatrix} \sum y_k^2 & \sum y_k \\ \sum y_k & (n+1) \end{pmatrix},$$

and

$$({}^tAA)^{-1} = \frac{1}{(n+1)\sum y_k^2 - (\sum y_k)^2}\begin{pmatrix} (n+1) & -\sum y_k \\ -\sum y_k & \sum y_k^2 \end{pmatrix},$$

and

$${}^tAB = \begin{pmatrix} y_0 \ldots y_n \\ 1 \ldots 1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum y_k x_k \\ \sum x_k \end{pmatrix},$$

then we have,

$$Y \neq ({}^tAA)^{-1}({}^tAB) = \frac{1}{(n+1)\sum y_k^2 - (\sum y_k)^2}\begin{pmatrix} (n+1)\sum y_k x_k - \sum y_k \sum x_k \\ -\sum y_k \sum y_k x_k + \sum y_k^2 \sum x_k \end{pmatrix}.$$

Proposition 2: The open subset Ω is the largest open connected and includes $(0, 0)$.

Proof: The subset Ω is open because it's the inverse image of the interval $(0, +\infty)$ by the continuous map $h: \mathbb{R}^2 \to \mathbb{R}$ described by $h(x, y) = 1 - x - ay$.

Proposition 3: The system (2) is of class C^∞ on the subset Ω.

Proof: Because the coordinates of the system (2) is polynomial.

Proposition 4: $Df(x, y)$ is an isomorphism for $\mathbb{R}^2$on $\mathbb{R}^2$ for all $(x, y) \in \Omega$.

Proof: Because, for all $(x, y) \in \Omega$. We have $detDf(x, y) = a(1 - x - ay) > 0$.

Proposition 5: The system (2) is one to one on Ω.

Proof: We use the following standard results: the system (2) is one to one if $f(x_1, y_1) = f(x_2, y_2)$; it implies that $(x_1 = x_2, y_1 = y_2)$. Let (x_1, y_1) and (x_2, y_2) with $1 - x_1 - ay_1 > 0$ and $1 - x_2 - ay_2 > 0$. In our case $f(x_1, y_1) = f(x_2, y_2)$ equivalent to $(x_1 - ay_1 = x_2 - ay_2, x_1(1 - ay_1) = x_2(1 - ay_2))$ or $(x_1 + (1 - ay_1) = x_2 + (1 - ay_2), x_1(1 - ay_1) = x_2(1 - ay_2))$ we observe that the two coordinates $(x_1, 1 - ay_1)$ and $(x_2, 1 - ay_2)$ have the same total and the same product, and it is convenient to distinguish two possibilities:

a) $x_1 = x_2$ and $y_1 = y_2$ in this case the map (2) is one to one on Ω.
b) $x_1 = 1 - ay_2$ and $x_2 = 1 - ay_1$, than from the choice of (x_1, y_1) we have $1 - x_2 - ay_2 = 1 - (1 - ay_1) - (1 - x_1) = -(1 - x_1 - ay_1) < 0$ this is impossible since $1 - x_2 - ay_2 > 0$. Finally, we conclude that the system (2) is one to one on Ω.

Proposition 6: $f(\Omega) = \{(X, Y) \in \mathbb{R}^2 : (1 + X)^2 - 4Y > 0\}$.

Proof: Let $(X, Y) \in \mathbb{R}^2$ we want to find the conditions that must be satisfied the coordinate (X, Y) in order to exist $(x, y) \in \Omega$ in which $f(x, y) = (X, Y)$. We have $f(x, y) = (X, Y)$ equivalent $(x - ay = X, x(1 - ay) = Y)$ or $(x + (1 - ay) = 1 + X, x(1 - ay) = Y)$. Therefore, x and $1 - ay$ are two solutions of the following equation of the variable T: $T^2 - (1 + X)T + Y = 0$ and the condition to accept this equation solutions in $\mathbb{R}$ is $\Delta = (1 + X)^2 - 4Y > 0$. If the last inequality satisfied, then we have:

$$\begin{cases} x = \frac{1+X-\sqrt{\Delta}}{2} \\ 1 - ay = \frac{1+X+\sqrt{\Delta}}{2}, \end{cases} \tag{3}$$

or

$$\begin{cases} x = \frac{1+X+\sqrt{\Delta}}{2} \\ 1 - ay = \frac{1+X-\sqrt{\Delta}}{2}. \end{cases} \tag{4}$$

The solution (3) is suitable because of $1 - x - ay = \sqrt{\Delta} > 0$ but the solution (4) is not suitable because it gives $1 - x - ay = -\sqrt{\Delta} < 0$ and also we reject the case $\Delta = 0$. Because it leads to $1 - x - ay = 0$. We conclude from the above discussion that:

$f(\Omega) = \{(X, Y) \in \mathbb{R}^2 : (1 + X)^2 - 4Y > 0\}$. We remark that the subset $f(\Omega)$ includes $A(0,0)$.

Proposition 7: $f_\Omega^{-1} : f(\Omega) \rightarrow \mathbb{R}^2$ given by:

$$\begin{cases} \bar{X} = \frac{1+X-\sqrt{\Delta}}{2} \\ \bar{Y} = \frac{1-X-\sqrt{\Delta}}{2a} \end{cases} \quad (5)$$

Proof: It's very easy work to verify this proposition by using (3).

Proposition 8: In the positive quadrant, all orbits of the system (2) are bounded.

Proof: It follows from the system (2) that $x_{n+1} \leq x_n$ and $y_{n+1} \leq x_n$ for all positive integer n since $x_n > 0, y_n > 0$ and $a > 0$, thus we have $x_{n+1} \leq x_n \leq x_{n-1} \leq \cdots \leq x_1 \leq x_0$ and $y_{n+1} \leq x_n \leq x_{n-1} \leq \cdots \leq x_1 \leq x_0$.

For all positive integer n. Then the sequences are monotone decreasing and so are bounded from above by x_0. It follows that the orbit of the system (2) is bounded.

3. Bifurcation properties of the system

The following section further investigates the dynamical behaviors of the chaotic system (2), including the stability of fixed point and bifurcations, Lyapunov exponents, bifurcation diagram, and Phases portraits.

3.1. Local stability conditions

The only fixed point of the system (2) is $A(0,0)$. The Jacobi matrix of the system (2) evaluated at a point (x, y) is given by:

$$Df(x,y) = \begin{pmatrix} 1 & -a \\ 1 - ay & -ax \end{pmatrix} \quad (6)$$

and $detDf(x, y) = a(1 - x - ay)$, at the fixed point $A(0,0)$, the Jacobi matrix is given by:

$$Df(0,0) = \begin{pmatrix} 1 & -a \\ 1 & 0 \end{pmatrix}$$

The characteristic polynomial of the Jacobi matrix of the system (2) calculated at the fixed point A, which takes the form: $P_A(\lambda) = \lambda^2 - \lambda + a$, according to the criterion available in Elaydi (1996), we conclude that the fixed point A of the system (2) is asymptotically stable if and only if the following conditions hold:

$$1 + 1 + a > 0, 1 - 1 + a > 0, 1 - a > 0$$

or, equivalently,

$$0 < a < 1$$

For example, if we choose $a = 0.7$ then with this value the fixed point A is asymptotically stable, and we have the following two eigenvalues $\lambda_1 = 0.50 - i0.67$ and $\lambda_2 = 0.50 + i0.67$ thus $|\lambda_{i(1\leq i\leq 2)}|<1$.

On the other hand, the local stability of A is studied by evaluating the eigenvalues of the Jacobi $J_{(0,0)}$.

If $a < \frac{1}{4}$ the eigenvalues of $J_{(0,0)}$ are $\lambda_1 = \frac{1-\sqrt{1-a4}}{2}$ and $\lambda_2 = \frac{1+\sqrt{1-a4}}{2}$. Then one has the following results:

(a) $|\lambda_1| < 1$ and $|\lambda_2| < 1$, if and only if $0 < a < \frac{1}{4}$, system (2) is attracting at this fixed point.
(b) $\lambda_1 = \lambda_2 = \frac{1}{4} < 1$, if and only if $a = \frac{1}{4}$, system (2) is attracting at this fixed point.
(c) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (impossible because $a > 0$), system (2) is not a saddle at this fixed point.
(d) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (impossible because $a > 0$), system (2) is not a saddle at this fixed point.
(e) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ (impossible because $a > 0$), system (2) is non-repelling at this fixed point.

If $a > \frac{1}{4}$ the eigenvalues of $J_{(0,0)}$ are $\lambda_1 = \frac{1-i\sqrt{-(1-a4)}}{2}$ and $\lambda_2 = \frac{1+i\sqrt{-(1-a4)}}{2}$. Then one has the following results:

(a) $|\lambda_1| = |\lambda_2| < 1$, if and only if $\frac{1}{4} < a < 1$, system (2) is attracting at this fixed point.
(b) $|\lambda_1| = |\lambda_2| = 1$, if and only if $a = 1$, system (2) is non-hyperbolic at this fixed point.
(c) $|\lambda_1| = |\lambda_2| > 1$, if and only if $a > 1$, system (2) is unstable at this fixed point.
(d) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (impossible because $a > 0$), system (2) is not a saddle at this fixed point.
(e) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (impossible because $a > 0$), system (2) is not a saddle at this fixed point.
(f) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ (impossible because $a > 0$), system (2) is non-repelling at this fixed point.

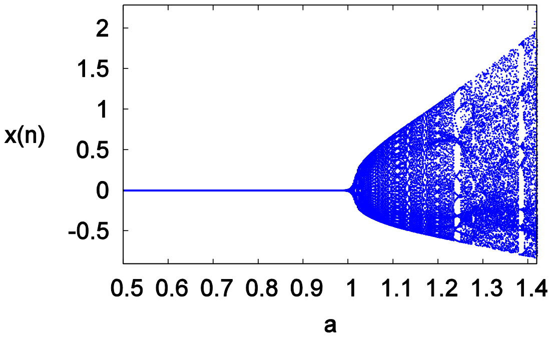
3.2. Numerical results

In this subsection, we will illustrate some observed chaotic attractors, the dynamical behaviors of the system (2) are investigated numerically. Fig. 1 shows the bifurcation diagram and the diagram of the variation of Lyapunov exponent of the system (2) by varying the parameter a. For the range $0.5 \leq a \leq 1.42$. It can be observed from Fig. 1 that system (2) undergoes the following dynamical behaviors as a increases:

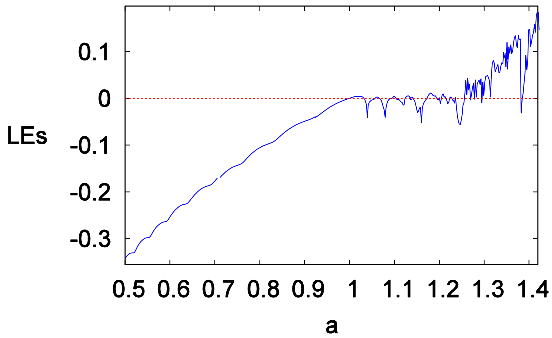
- For $0.5 \leq a < 1$, system (2) is a fixed point.
- For $a = 1$, the fixed point A loses stability at $a = 1$, and we have the following two eigenvalues $\lambda_1 = \frac{1+i\sqrt{3}}{2}$ and $\lambda_2 = \frac{1-i\sqrt{3}}{2}$, thus $|\lambda_{i(1\leq i\leq 2)}|$=1. At this value, a Hopf bifurcation occurs, and via the quasi-periodic route to chaos, chaotic behavior can be observed. Fig. 1a shows a diagram description of this scenario of chaos.
- For $0.5 < a \leq 1.42$, system (2) is chaotic via the quasi-periodic route to chaos, and there are several

quasi-periodic windows. If we fix the parameter a to the value $a = 1.40$ at the point, the dynamical behavior of the system (2) is chaotic, which is verified by the corresponding largest Lyapunov exponent is positive, as shown in Fig. 1b. The corresponding chaotic attractor is shown in Fig. 2d.

Also, Fig. 2a shows Quasi-periodic orbit of the system (2) ($a = 1.25$), Fig. 2b shows Chaotic attractor of the system (2) ($a = 1.27$), and Fig. 2c shows Chaotic attractor of the system (2) ($a = 1.28$).

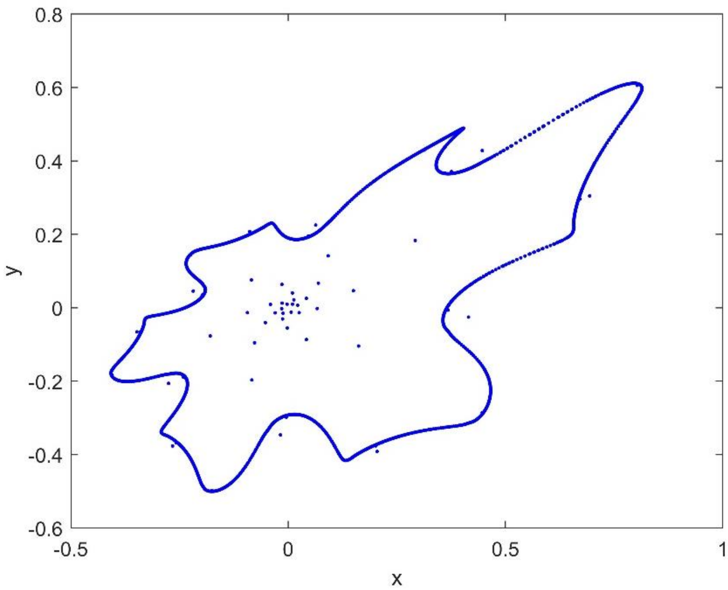


a: Bifurcation diagram of the system (2) versus $0.5 \leq a \leq 1.42$

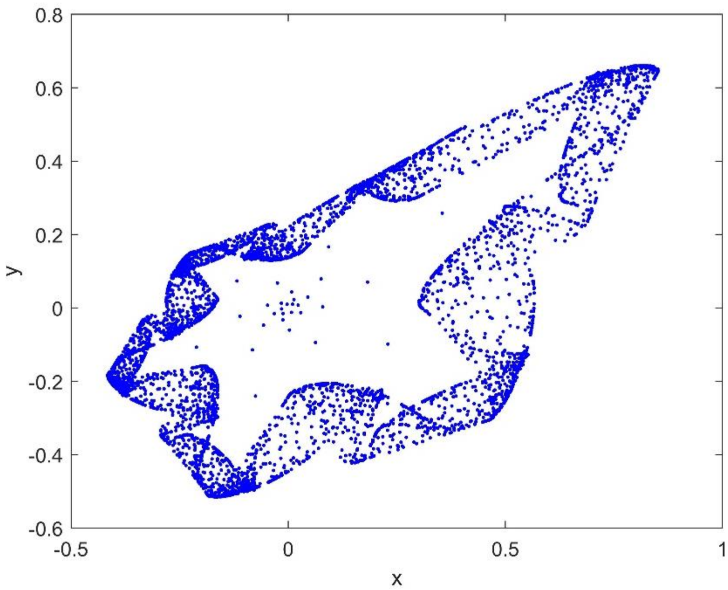


b: Lyapunov exponent of the system (2) versus $0 \leq a \leq 1.4$

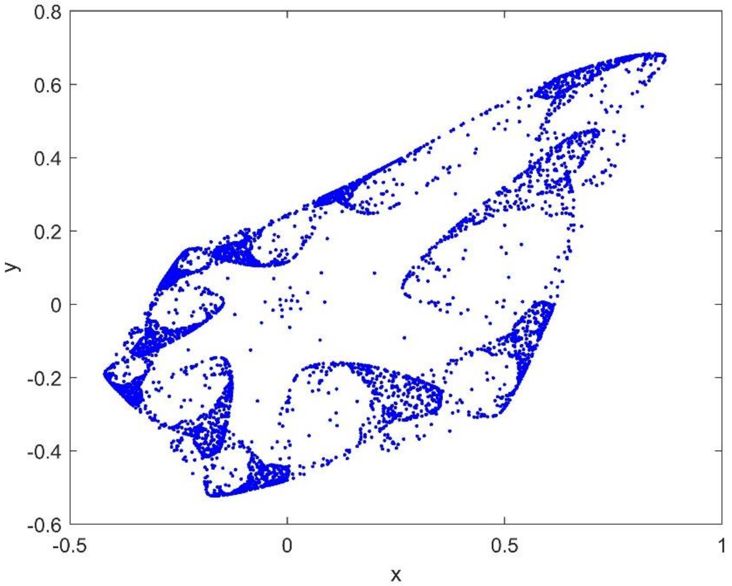
Fig. 1: Bifurcation diagram and Lyapunov exponent of the system (2)



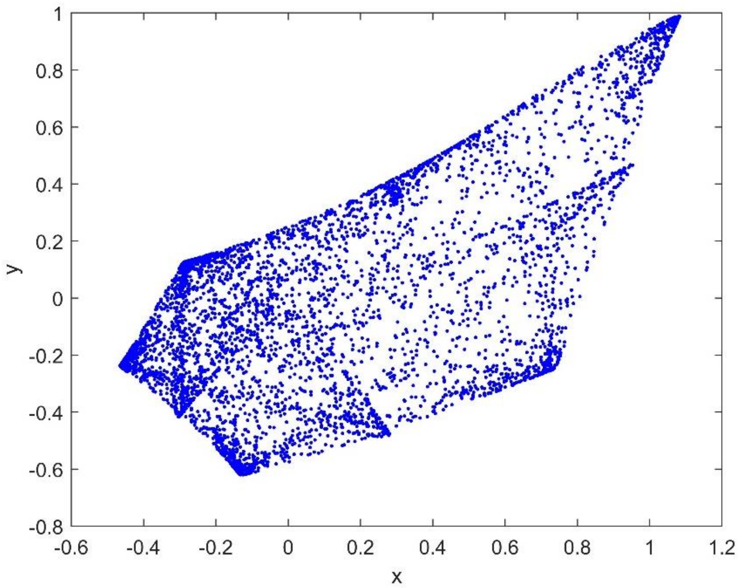
a: Quasi-periodic orbit of the system (2) ($a = 1.25$)



b: Chaotic attractor of the system (2) ($a = 1.27$)



c: Chaotic attractor of the system (2) ($a = 1.28$)



d: Chaotic attractor of the system (2) ($a = 1.40$)

Fig. 2: Quasi-periodic orbit, chaotic attractor and chaotic attractor of the system (2)

4. Conclusion

This paper is devoted to the rigorous proof of the existence of some interesting dynamical properties of the system using the standard methods available in most kinds of literature on analysis mathematics. Also, the dynamics of the system are described numerically in some detail.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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