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Dynamical study of the class of difference equation $x_{n+1} = x_{n-2q+1}/A + Bx_{n-2q+1}x_{n-q+1}$

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ABSTRACT

This work is devoted to present a study of the class of the difference equations $x_{n+1} = x_{n-2q+1}/A + Bx_{n-2q+1}x_{n-q+1}$, $q = 1,2, \ldots$ with arbitrary initial data, where A and B are arbitrary parameters, and q is an arbitrary nonnegative integer. We present a detailed investigation of the behavior of the solution, including their dependence on parameters and initial conditions. Local and global stabilities of the equilibrium points are discussed. The existence of a periodic solution is studied. Numerical simulations are given to assure the correctness of the analytical results. This study improves and surpasses studies of several forms of difference equations that have been investigated earlier by many researchers.

1. Introduction

Mathematics Difference equations play an important role in describing dynamical systems and presenting many numerical schemes (Agiza and Elsadany, 2004; Ahmed et al., 2015; Ahmed and Hegazi, 2006; Askar, 2014; Elabbasy et al., 2014; El-Metwally et al., 2015; El-Morshedy and Liz, 2005; Elsadany, 2010; Elsadany et al., 2013; Elsadany and Matouk, 2014; Karatas et al., 2006; Matouk et al., 2015). Many applications of difference equations can be found in various fields of science such as game theory, mathematical biology, physics, and engineering (Ahmed et al., 2015; Ahmed and Hegazi, 2006; Askar et al., 2016; Elabbasy et al., 2014; El-Morshedy and Liz, 2006; Elsadany et al., 2013; Elsadany and Matouk, 2014; Wang et al., 2017a). Because of these applications, many researchers focused on studying difference equations. For some of these studies, we refer the reader to Cinar (2004a, 2004b) and Elsayed (2008, 2009a, 2009b, 2009c, 2009d, 2010a, 2010b, 2010c). In recent years, many researchers investigated the qualitative behavior of nonlinear rational difference equations and systems of difference equations.

Elabbasy et al. (2007) discussed the global stability character of the difference equation $x_{n+1} = \frac{Cx_n+\beta x_{n-1}+\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}}$. Amleh et al. (2001) studied the third order rational difference equation $x_{n+1} = \frac{a+bx_{n-1}}{A+Bx_{n-2}}$. Aloqeili (2006) obtained the solution of the difference equation $x_{n+1} = \frac{x_{n-1}}{a-x_n x_{n-1}}$. Cinar (2004a) gave the solution for the difference equation $x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}$. Motivated by the work of Cinar (2004b) and others, Wang et al. (2010) studied the asymptotic behavior of the solutions for the difference equation $x_{n+1} = \frac{\sum_{i=1}^{l} A_{s_i} x_{n-s_i}}{B+C\prod_{j=1}^{k} x_{n-t_i}} + Dx_n$, where the parameters A_{s_i}, B, C, and D are positive real numbers. Their technique is based on a variational iteration method. They introduce the notion of mixed monotone property of functions. Then they use it to obtain an interesting result [54, Theorem 3.2]. This result gives sufficient conditions that grantee that the equation has a unique equilibrium point, and this equilibrium point is globally attractor. Using a similar technique, Wang et al. (2011) investigated the asymptotic behavior of the solutions for the difference equation $x_{n+1} = \frac{\sum_{i=1}^{l} A_{s_i} x_{n-s_i}}{B+C\prod_{j=1}^{k} x_{n-t_i}}$, where the parameters A_{s_i}, B, and C, are positive real numbers. Wang et al. (2016) studied the dynamical behavior and determined the expressions of the solutions for a system of two rational difference equations $x_{n+1} = \frac{x_{n-3}}{A+x_{n-3}y_{n-1}}$, $y_{n+1} = \frac{y_{n-3}}{B+y_{n-3}x_{n-1}}$. Very recently, Wang et al. (2017b) and

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Liu et al. (2019) also used variational iteration techniques to described the asymptotic behavior of the equilibrium points of the systems of difference equations $x_{n+1}=\frac{x_{n-1}x_{n-2}}{A+By_{n-3}}$, $y_{n+1}=\frac{y_{n-1}y_{n-2}}{C+Dx_{n-3}}$, $x_{n+1}=\frac{x_{n-3}-y_{n-1}}{A+x_{n-3}y_{n-1}}$, and $y_{n+1}=\frac{y_{n-3}-x_{n-1}}{A+y_{n-3}x_{n-1}}$, respectively. For more recent studies of systems of rational difference equations (Haddad et al. 2017a; 2017b; 2018).

Elsayed (2011a; 2011b; 2011c) obtained the solutions for the following difference equations $x_{n+1}=\frac{x_{n-3}}{\pm1\pm x_{n-1}x_{n-3}}$, $x_{n+1}=\frac{x_{n-9}}{\pm1\pm x_{n-4}x_{n-9}}$. Then later, Elsayed et al. (2017) generalized this work, where they gave a detailed analytical study and behavior of the solutions of $x_{n+1}=\frac{\alpha x_{n-3}}{A+Bx_{n-1}x_{n-3}}$ and investigated many of its properties such as local stability and global attractivity of its equilibrium points. Ghazel et al. (2017) considered the difference equations $x_{n+1}=\frac{Cx_{n-5}}{A+Bx_{n-2}x_{n-5}}$, where they obtained the solution of this equation, investigated its asymptotic behavior, determined its forbidden set, and discussed the existence of periodic solutions. Wang et al. (2017c) studied the max-type difference equation $x_{n+1}=\max\left\{\frac{A}{x_n},\frac{A}{x_{n-1}},x_{n-2}\right\}$. Later Wang et al. (2018) investigated the boundedness and asymptotic behavior of systems of max-type difference equation. Their work generalizes and improves many results concerning systems of max-type non-linear difference equations. The sufficient conditions obtained in their paper provide flexibility for applications and analysis of such systems.

The aim of this paper is to give a great generalization to the study of the qualitative behavior of nonlinear rational difference equations. We consider a general class of difference equations of the form:

$$x_{n+1}=\frac{x_{n-2q+1}}{A+Bx_{n-2q+1}x_{n-q+1}}, \tag{1}$$

where A and B are arbitrary constants and with arbitrary initial data $x_{-2q+1}=a_{-2q+1}$, $x_{-2q+2}=a_{-2q+2}, \ldots, x_0=a_0$. We give a detailed analytical study of this difference equation. Where we obtain the solution of this equation and investigate its convergence. We also investigate its asymptotic behavior, determine its forbidden set, and discuss the existence of its periodic solutions. The order of this difference equation, namely $2q$, is kept as an arbitrary parameter. This allows us to make a significant contribution to the study of difference equations. We were able to generalize and improve results about many forms of difference equations such as the ones studied in Aloqeili (2006), Cinar (2004a), Elsayed (2011a, 2011b, 2011c), and Ghazel et al. (2017). For instance, taking $q=1$, $A=1/a$, and $B=b/a$ in Eq. 1, yields the second order difference equation that was considered in Cinar (2004b). On the other hand, substituting $q=3$ and replacing A and B by A/C and B/C respectively produce the sixth order difference equation considered in Ghazel et al. (2017).

In the next section, we introduce some basic definitions and primary results that will be needed in later sections. Section 3, discusses the equilibrium points and their stability. We show that Eq. 1 has either one equilibrium point, namely $\bar{x}=0$, or three equilibrium points, $\bar{x}=0$, $\mp\sqrt{(1-A)/B}$. The stability of these equilibrium points is also investigated. Theorem (4.1) gives complete analytical expressions for the solutions of Eq. 1, and the good sets are described in Theorem (4.2). In Section 5, we present a detailed study for the convergence of the solutions of Eq. 1, and the periodicity of these solutions is dealt with in Section 6. In the last section, some numerical simulations are given to support the theoretical results.

2. Preliminaries

A difference equation of order k is an equation of the form:

$$x_{n+1}=F(x_n,x_{n-1},\cdots,x_{n-(k-1)}),\ n=0,1,2,. \tag{2}$$

where $F=F(u_0,u_1,\cdots,u_{k-1})$ is a function that maps some set I^k into I. The set I is usually an interval of real numbers or a union of intervals. A solution of Eq. 2 is a sequence $(x_n)_{n\geq -k+1}$ that satisfies Eq. 2 for all $n\geq 0$. The vector $v_0(x)=(x_0, x_{-1}, \ldots, x_{-k+1})\in I^k$ is called the vector of initial conditions associate with the solution $(x_n)_{n\geq -k+1}$ of Eq. 2.

A point $\bar{x}\in R$ is called an equilibrium point of Eq. 2, if $\bar{x}=F(\bar{x},\bar{x},\cdots,\bar{x})$. If the function F is continuously differentiable in some open neighborhood of $\bar{x}$, then the linearized equation of Eq. 2 about $\bar{x}$ is given by:

$$y_{n+1}=\sum_{p=0}^{k-1}\frac{\partial F}{\partial u_p}(\bar{x},\bar{x},\cdots,\bar{x})y_{n-p}, \tag{3}$$

and the characteristic equation of Eq. 3 about $\bar{x}$ is given by:

$$\lambda^k-\sum_{p=0}^{k-1}\frac{\partial F}{\partial u_p}(\bar{x},\bar{x},\cdots,\bar{x})\lambda^{k-1-p}=0. \tag{4}$$

In the sequel, the norm of a vector $u\in I^k$ is defined as $\|u\|=\sum_{i=0}^{k-1}|u_i|$, and for an equilibrium point $\bar{x}\in I$ we denote, by $v(\bar{x})\in I^k$, the vector $v(\bar{x})=(\bar{x},\bar{x},\ldots,\bar{x})$.

Definition 2.1 (Stability):

1. An equilibrium point $\bar{x}$ of Eq. 2 is called locally stable if, for every $\varepsilon>0$, there exists $\delta>0$ such that, if $(x_n)_{n\geq -k+1}$ is a solution of Eq. 2 with $|v_0(x)-v(\bar{x})|<\delta$, then $|x_n-\bar{x}|<\varepsilon$, for all $n\geq 0$. If $\bar{x}$ is not locally stable, then it is called unstable.
2. An equilibrium point $\bar{x}$ of Eq. 2 is called locally asymptotically stable if it is locally stable, in addition there exists $\gamma>0$ such that $lim_n\, x_n=\bar{x}$ for all solutions $(x_n)_{n\geq -k+1}$ of Eq. 2 with $|v_0(x)-v(\bar{x})|<\delta$.
3. If $lim_n\, x_n=\bar{x}$ for all solutions $(x_n)_{n\geq -k+1}$ of Eq. 2, then $\bar{x}$ is called a global attractor.

4. If $\bar{x}$ is both locally stable and global attractor, then it is called globally asymptotically stable.

Definition 2.2: Let $\bar{x}$ be an equilibrium point of Eq. 2 and let Λ be the set of all roots of the characteristic Eq. 4 about $\bar{x}$. Then:

1. $\bar{x}$ is said to be hyperbolic if $|\lambda| \neq 1$ for all $\lambda \in \Lambda$, otherwise it is called nonhyperbolic;
2. $\bar{x}$ is called saddle if there are two roots $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_1 < 1$ and $\lambda_2 > 1$;
3. $\bar{x}$ is called repeller if $|\lambda| > 1$ for all $\lambda \in \Lambda$.

The next theorem, obtained by Kocic and Ladas (1993), is called the Linearized Stability Theorem.

Theorem 2.1: Let $\bar{x}$ be an equilibrium point of Eq. 2 and let Λ be the set of all roots of the characteristic Eq. 4 about $\bar{x}$.

1. If $\lambda < 1$, for all $\lambda \in \Lambda$, then $\bar{x}$ is locally asymptotically stable.
2. If $\lambda > 1$, for some $\lambda \in \Lambda$, then $\bar{x}$ is unstable.

Definition 2.3: A solution $(x_n)_{n\geq -k+1}$ of Eq. 2 is said to be periodic with period p or a periodic$-p$ solution if:

$$x_{n+p} = x_n, \forall n \geq -k + 1. \tag{5}$$

A solution is called periodic with prime period p if p is the smallest positive integer for which Eq. 5 holds.

3. Stability analysis of the equilibrium points

In this section, we discuss the nature of the equilibrium points of the difference equation given by

$$x_{n+1} = F(x_n, x_{n-1}, \cdots, x_{n-2q+1}),, \tag{6}$$

where F is the continuous function defined on $\mathbb{R}^{2q}$ as $F(u_0, u_1, \cdots, u_{2q-1}) = \frac{u_{2q-1}}{A+Bu_{2q-1}u_{q-1}}$

Theorem 3.1: Let $(x_n)_{n\geq -2q+1}$ be a solution of Eq. 6.

1. If $B(A-1) \geq 0$, then Eq. 6 has a unique equilibrium point $\bar{x}_1 = 0$.
2. If $B(A-1) < 0$, then Eq. 6 has exactly three equilibrium points, namely $\bar{x}_1 = 0$, $\bar{x}_2 = \sqrt{(1-A)/B}$, and $\bar{x}_3 = -\sqrt{(1-A)/B}$.

Proof: The result follows directly from the fact that the equilibrium points of Eq. 6 are the real roots of the equation $\bar{x}(B\bar{x}^2 - A + 1) = 0$.

The partial derivatives of the function F with respect to $u_i, i = 0, 1, \ldots, 2q-1$ are $\frac{\partial F}{\partial u_{q-1}} = \frac{-Bu_{2q-1}^2}{(A+Bu_{q-1}u_{2q-1})^2}$, $\frac{\partial F}{\partial u_{2q-1}} = \frac{A}{(A+Bu_{q-1}u_{2q-1})^2}$, and zero otherwise. Thus, the characteristic equation of the linearized equation of Eq. 6 about an equilibrium point $\bar{x}$ can be simplified to

$$\lambda^{2q} + \frac{B\bar{x}^2}{(A+B\bar{x}^2)^2}\lambda^p - \frac{A}{(A+B\bar{x}^2)^2} = 0. \tag{7}$$

Theorem 3.2: Let $(x_n)_{n\geq -2q+1}$ be a solution of Eq. 6, and let $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ be the equilibrium points given in Theorem (3.1).

1. If $|A| < 1$, then $\bar{x}_1$ is a repeller.
2. If $|A| = 1$, then $\bar{x}_1$ is nonhyperbolic.
3. If $|A| > 1$, then $\bar{x}_1$ is locally asymptotically stable.
4. If $B(A-1) < 0$, then the equilibrium points $\bar{x}_2$ and $\bar{x}_3$ are nonhyperbolic. Moreover, they are unstable provided that $|A| > 1$.

Proof: From Eq. 7 it results in the characteristic equation about $\bar{x}_1 = 0$ is:

$$\lambda^{2q} - \frac{1}{A} = 0. \tag{8}$$

The roots of this equation satisfy $|\lambda|^{2q} = 1/A$. This proves 1, 2, and 3. Now, assume that $B(A-1) < 0$. Then the characteristic equation about $\bar{x}_2$, as well as $\bar{x}_3$, can be written as,

$$(\lambda^q + 1)(\lambda^q - A) = 0. \tag{9}$$

Thus $|\lambda| = 1$ or $|\lambda| = A^{1/q}$. So $\bar{x}_2$ and $\bar{x}_3$ are nonhyperbolic, and if $|A| > 1$, then they are unstable.

4. Analytical expressions of $(x_n)_n$

In this section, we give some analytical expressions of the sequence $(x_n)_{n\geq -2q+1}$.

Theorem 4.1: Let $(x_n)_{n\geq -2q+1}$ be a sequence given by Eq. 1. Then for each $r \in \{2q-1, 2q-2, \ldots, q\}$ and all $n \geq 0$, we have:

$$x_{2qn-r} = a_{-r}\frac{\prod_{p=1}^{n-1}(A^{2p} + Ba_{-r}a_{-r+q}\sum_{k=0}^{2p-1}A^k)}{\prod_{p=0}^{n-1}(A^{2p+1} + Ba_{-r}a_{-r+q}\sum_{k=0}^{2p}A^k)} \tag{10}$$

and

$$x_{2qn-r+q} = a_{-r+q}\prod_{p=0}^{n-1}\left(\frac{A^{2p+1} + Ba_{-r}a_{-r+q}\sum_{k=0}^{2p}A^k}{A^{2p+2} + Ba_{-r}a_{-r+q}\sum_{k=0}^{2p+1}A^k}\right). \tag{11}$$

Proof: Fix $r \in \{2q-1, 2q-2, \ldots, q\}$, and let $Q_p = A^p + Ba_{-r}a_{-r+q}\sum_{k=0}^{p-1}A^k$, where p is any natural number.

Consequently, we need to show that $x_{2qn-r} = a_{-r}\frac{\prod_{p=1}^{n-1}Q_{2p}}{\prod_{p=0}^{n-1}Q_{2p+1}}$ and $x_{2qn-r+q} = a_{-r+q}\prod_{p=0}^{n-1}\left(\frac{Q_{2p+1}}{Q_{2p+2}}\right)$. We will do this by induction on n. It is evident that the results hold for $n = 0$. Let $n \geq 0$ be an integer, and suppose that the results hold for all nonnegative integers $k \leq n$. We shall now prove that the identities hold for the step $n+1$.

$$x_{2q(n+1)-r} = \frac{x_{2qn-r}}{A+B\,x_{2qn-r+q}x_{2qn-r}} =$$
$$\frac{a_{-r}\left(\prod_{p=1}^{n-1}Q_{2p}\right)/\left(\prod_{p=0}^{n-1}Q_{2p+1}\right)}{A+Ba_{-r}a_{-r+q}\prod_{p=0}^{n-1}\left(\frac{Q_{2p+1}}{Q_{2p+2}}\right)\left(\prod_{p=1}^{n-1}Q_{2p}\right)/\left(\prod_{p=0}^{n-1}Q_{2p+1}\right)} =$$
$$\frac{a_{-r}\prod_{p=1}^{n-1}Q_{2p}\prod_{p=1}^{n}Q_{2p}}{A\prod_{p=1}^{n}Q_{2p}\prod_{p=0}^{n-1}Q_{2p+1}+Ba_{-r}a_{-r+q}\prod_{p=0}^{n-1}Q_{2p+1}\prod_{p=1}^{n-1}Q_{2p}} =$$
$$\frac{a_{-r}\prod_{p=1}^{n}Q_{2p}}{\left(AQ_{2n}+Ba_{-r}a_{-r+q}\right)\prod_{p=0}^{n-1}Q_{2p+1}} =$$
$$\frac{a_{-r}\prod_{p=1}^{n}Q_{2p}}{\left(A\left(A^{2n}+B\,a_{-r}a_{-r+q}\sum_{k=0}^{2n-1}A^k\right)+Ba_{-r}a_{-r+q}\right)\prod_{p=0}^{n-1}Q_{2p+1}} =$$
$$a_{-r}\frac{\prod_{p=1}^{n}Q_{2p}}{\prod_{p=1}^{n}Q_{2p+1}}$$

similarly,

$$x_{2q(n+1)-r+q} = \frac{x_{2qn-r+q}}{A+B\,x_{2q(n+1)-r}x_{2qn-r+q}} =$$
$$\frac{a_{-r+q}\prod_{p=0}^{n-1}\left(\frac{Q_{2p+1}}{Q_{2p+2}}\right)}{A+Ba_{-r}a_{-r+q}\prod_{p=0}^{n-1}\left(\frac{Q_{2p+1}}{Q_{2p+2}}\right)\left(\prod_{p=1}^{n}Q_{2p}\right)/\left(\prod_{p=0}^{n}Q_{2p+1}\right)} =$$
$$\frac{a_{-r+q}\prod_{p=0}^{n-1}Q_{2p+1}}{\left(\prod_{p=0}^{n-1}Q_{2p+1}\right)\left(AQ_{2n+1}\prod_{p=0}^{n-1}Q_{2p+2}+Ba_{-r}a_{-r+q}\prod_{p=1}^{n}Q_{2p}\right)/\left(\prod_{p=0}^{n}Q_{2p+1}\right)} =$$
$$\frac{a_{-r+q}\prod_{p=0}^{n}Q_{2p+1}}{\left(AQ_{2n+1}+Ba_{-r}a_{-r+q}\right)\prod_{p=1}^{n}Q_{2p}} =$$
$$\frac{a_{-r+q}\prod_{p=0}^{n}Q_{2p+1}}{\left(A\left(A^{2n+1}+B\,a_{-r}a_{-r+q}\sum_{k=0}^{2n}A^k\right)+Ba_{-r}a_{-r+q}\right)\prod_{p=1}^{n}Q_{2p}} =$$
$$a_{-r+q}\frac{\prod_{p=0}^{n}Q_{2p+1}}{Q_{2n+2}\prod_{p=1}^{n}Q_{2p}}.$$

This completes the proof.

Next, we obtain simplified expressions of Eq. 10 and Eq. 11 when $A = 1$ and $A \neq 1$.

Corollary 4.1: Let $(x_n)_{n\geq -2q+1}$ be a sequence defined by Eq. 1 with $A = 1$. Then for $r = 2q-1, 2q-2, \ldots, q$,

$$x_{2qn-r} = \frac{a_{-r}\prod_{p=1}^{n-1}(1+2pBa_{-r}a_{-r+q})}{\prod_{p=0}^{n-1}(1+(2p+1)Ba_{-r}a_{-r+q})} \tag{12}$$

and for $s = q-1, q-2, \ldots, 0$,

$$x_{2qn-s} = a_{-s}\prod_{p=0}^{n-1}\left(\frac{1+(2p+1)Ba_{-s-q}a_{-s}}{1+(2p+2)Ba_{-s-q}a_{-s}}\right) \tag{13}$$

Corollary 4.2: Let $(x_n)_{n\geq -2q+1}$ be a sequence defined by Eq. 1 with $A \neq 1$. Then for $r = 2q-1, 2q-2, \ldots, q$,

$$x_{2qn-r} = \frac{a_{-r}(1-A)\prod_{p=1}^{n-1}\left(Ba_{-r}a_{-r+q}-(A-1+Ba_{-r}a_{-r+q})A^{2p}\right)}{\prod_{p=0}^{n-1}\left(Ba_{-r}a_{-r+q}-(A-1+Ba_{-r}a_{-r+q})A^{2p+1}\right)}, \tag{14}$$

and for $s = q-1, q-2, \ldots, 0$,

$$x_{2qn-s} = a_{-s}\prod_{p=1}^{n-1}\left(\frac{Ba_{-s-q}a_{-s}-(A-1+Ba_{-s-q}a_{-s})A^{2p+1}}{Ba_{-s-q}a_{-s}-(A-1+Ba_{-s-q}a_{-s})A^{2p}}\right). \tag{15}$$

Proof: Since $A \neq 1$, applying the binomial identity yields that

$$\sum_{k=0}^{2q-1}A^k = \frac{1-A^{2q}}{1-A}, \tag{16}$$
$$\sum_{k=0}^{2q}A^k = \frac{1-A^{2q+1}}{1-A}, \tag{17}$$

and

$$\sum_{k=0}^{2q+1}A^k = \frac{1-A^{2q+2}}{1-A}. \tag{18}$$

Now Eqs. 14 and 15 can be obtained directly by substituting Eqs. 16, 17, and 18 in Eqs. 10 and 11.

The good set of a difference equation is the set of conditions that any initial data must satisfy so that the associated solution of the difference equation is defined for all natural number n. The complement of the good set is usually called the forbidden set. Determining good sets is a problem of great importance in the study of difference equations, and the interest in this problem has increased recently. For general information on good sets and forbidden sets of difference equations, we refer the reader to Azizi (2012), Grove and Ladas (2005), Kocic and Ladas (1993), Kulenovic and Ladas (2002), and Rubió-Masségú (2009). The next theorem describes the good set of the difference Eq. 1.

Theorem 4.2: The good set of the difference Eq. 1 is:

$$G = \left\{(a_{-r})_{0\leq r\leq 2q-1} \in \mathbb{R}^{2q} \,\middle|\, a_{-r}a_{-r-q} \notin \left\{\frac{-A^{p+1}}{B\sum_{k=0}^{p}A^k},\ p\geq 0\right\}, \forall r = q-1, \ldots, 0\right\} \tag{19}$$

More precisely, if $A \neq 1$, then,

$$G = \left\{(a_{-r})_{0\leq r\leq 2q-1} \in \mathbb{R}^{2q} \,\middle|\, a_{-r}a_{-r-q} \notin \left\{\frac{(A-1)A^p}{B(1-A^p)},\ p\geq 1\right\}, \forall r = 2q-1, \ldots, q\right\} \tag{20}$$

and, if $A = 1$, then,

$$G = \left\{(a_{-r})_{0\leq r\leq 2q-1} \in \mathbb{R}^{2q} \,\middle|\, a_{-r}a_{-r-q} \notin \left\{\frac{-1}{Bp},\ p\geq 1\right\}, \forall r = 2q-1, \ldots, q\right\}. \tag{21}$$

Proof: Let $(x_n)_{n\geq -2q+1}$ be a sequence defined by Eq. 1 with initial data $x_{-2q+1} = a_{-2q+1}$, $x_{-2q+2} = a_{-2q+2}, \ldots, x_0 = a_0$. Then Eqs. 10 and 11 and the steps in the proof of Theorem (4.1) imply that (x_n) is defined for all integer $n \geq -2q+1$ if and only if $Q_p = A^p + B\,a_{-r}a_{-r+q}\sum_{k=0}^{p-1}A^k \neq 0$ for all p. So $A^{2p+1} + Ba_{-r}a_{-r+q}\sum_{k=0}^{2p}A^k \neq 0$ and $A^{2p+2} + B\,a_{-r}a_{-r+q}\sum_{k=0}^{2p+1}A^k \neq 0$, and hence $a_{-r}a_{-r+q} \neq -A^{p+1}/\left(B\sum_{k=0}^{p}A^k\right)$. Moreover, if $a_{-r}a_{-r+q} \neq -A^{p+1}/\left(B\sum_{k=0}^{p}A^k\right)$ for all $r \in \{2q-1, \ldots, 0\}$ and all p, then $Q_p = A^p + B\,a_{-r}a_{-r+q}\sum_{k=0}^{p-1}A^k \neq 0$ for all p, and so (x_n) is defined for all integer $n \geq -2q+1$. Similarly, the good sets in (20) and (21) are obtained directly from Corollaries (4.1) and (4.2), respectively

5. Convergence

In this section, we study the asymptotic behavior of a solution of difference Eq. 1.

Theorem 5.1: (*The case* $|A| < 1$) Let $(x_n)_{n\geq -2q+1}$ be a solution of Eq. 1. If $|A| < 1$, then for each $r \in$

$\{0,1,\ldots,2q-1\}$, the subsequence $(x_{2qn-r})_{n\geq 0}$ converges.

Proof: We divide the proof into two cases.

Case 1: $r \in \{2q-1,\ldots,q\}$.
If $A-1+Ba_{-r}a_{-r+q}=0$, then $(x_{2qn-r})_{n\geq 0}$ is constant, and hence it converges. Now, assume that $A-1+Ba_{-r}a_{-r+q}\neq 0$. Since $|A|\neq 1$, we can write Eq. 14 as $x_{2qn-r} = \frac{a_{-r}}{Ba_{-r}a_{-r+q}+A}\prod_{p=1}^{n-1}U_p$, where $U_p = \frac{1-\alpha A^{2p}}{1-\alpha A^{2p+1}}$ and $\alpha = \frac{A-1+Ba_{-r}a_{-r+q}}{Ba_{-r}a_{-r+q}}$. The Taylor expansion of U_p gives that $U_p = 1+\alpha(A-1)A^{2p}+o(A^{2p})$, which yields $U_p \sim 1+\alpha(A-1)A^{2p}$. Now depending on the sign of α, we can choose an integer N sufficiently large so that either $U_p>1$, for all $p\geq N$, or $0<U_p<1$, for all $p\geq N$. Since $\prod_{p\geq 1}(1+\alpha(A-1)A^{2p})$ converges, by equivalence criterion, it follows that $\prod_{p\geq 1}U_p$ converges. Therefore $(x_{2qn-r})_{n\geq 0}$ converges.

Case 2: $s \in \{q-1,\ldots,0\}$.
If $A-1+Ba_{-s-q}a_{-s}=0$, then again $(x_{2qn-s})_{n\geq 0}$ is constant, and hence it converges. Now assume that $A-1+Ba_{-s-q}a_{-s}\neq 0$. Then Eq. 15 can be written as $x_{2qn-s} = a_{-s}\prod_{p=1}^{n-1}V_p$, where $V_p = \frac{1-\beta A^{2p+1}}{1-\beta A^{2p}}$ and $\beta = \frac{A-1+Ba_{-s-q}a_{-s}}{Ba_{-s-q}a_{-s}}$. Similar to the argument in Case 1, we obtain that $V_p \sim 1+\beta(1-A)A^{2p}$. Also depending on the sign of β, we can find an integer N so that either $V_p>1$, for all $p\geq N$, or $0<V_p<1$, for all $p\geq N$. Since $\prod_{p\geq 1}(1+\beta(1-A)A^{2p})$ converges, by equivalence criterion $\prod_{p\geq 1}V_p$ converges, and hence the subsequence $(x_{2qn-s})_{n\geq 0}$ converges.

Remark 5.1: In Theorem (5.1), we have shown that if $|A|<1$, then every subsequence $(x_{2qn-r})_{n\geq 0}$, $r\in\{2q-1,\ldots,0\}$ converges to a real number l_r. A natural question that arises here is whether the whole sequence $(x_n)_{n\geq -2q+1}$ converges? We will see that $(x_n)_{n\geq -2q+1}$ maybe divergent. Indeed, consider the subsequences $(x_{2qn-r})_{n\geq 0}$ and $(x_{2qn-r-q})_{n\geq 0}$ for some $r=0,1,\ldots,q-1$. These subsequences are related by the equation,

$$x_{2q(n+1)-r-q} = \frac{x_{2qn-r-q}}{A+B\,x_{2qn-r-q}x_{2qn-r}} \quad (22)$$

Taking the limits on n in Eq. 22, yields $l_{-r-q} = \frac{l_{-r-q}}{A+B\,l_{-r}l_{-r-q}}$. Now $l_{-r-q}\neq 0$, otherwise it will contradict the fact that $\prod_{p\geq 1}U_p$ converges. So $l_{-r}l_{-r-q} = \frac{1-A}{B}$. Thus if $B<0$, then $l_{-r-q}\neq l_{-r}$, and so the sequence $(x_n)_{n\geq -2q+1}$ diverges. The numerical example (Fig. 1) illustrates that l_{-r-q} and l_{-r} are not necessarily equal in the case where $|A|<1$, even if we choose $B>0$.

Theorem 5.2: (*The case* $A=-1$) Let $(x_n)_{n\geq -2q+1}$ be a solution of Eq. 1. If $A=-1$, then the following statements are true.

1. For each $r=2q-1,2q-2,\ldots,q$, the subsequence (x_{2qn-r}) converges if and only if $a_{-r}a_{-r+q}\in(-\infty,min(0,2/B))\cup(max(0,2/B),\infty)\cup\{2/B\}$.
2. For each $r=q-1,q-2,\ldots,0$, the subsequence (x_{2qn-r}) converges if and only if $a_{-r}a_{-r-q}\in(min(0,2/b),max(0,2/B))\cup\{2/B\}$.

Proof:

1. Let $r\in\{2q-1,2q-2,\ \ldots\ ,q\}$. Then replacing A in Eq. 14 by -1, gives $x_{2qn-r} = \frac{a_{-r}}{(Ba_{-r}a_{-r+q}-1)^{n}}$. Therefore, $(x_{2qn-r})_n$ converges $\Leftrightarrow$ $|Ba_{-r}a_{-r+q}-1|>1$ or $Ba_{-r}a_{-r+q}-1=1$ $\Leftrightarrow$ $a_{-r}a_{-r+q}\in(-\infty,min(0,2/B))\cup(max(0,2/B),\infty)\cup\{2/B\}$.
2. Let $r\in\{q-1,q-2,\ldots,0\}$. Similarly, replacing A in Eq. 15 by -1, yields $x_{2qn-r} = a_{-r}(Ba_{-r}a_{-r-q}-1)^{n-1}$. So $(x_{2qn-r})_n$ converges if and only if $a_{-r}a_{-r-q}\in(min(0,2/b),max(0,2/b))\cup\{2/b\}$.

The following remark is deduced from the computations in the proof of Theorem (5.2).

Remark 5.2: Let $(x_n)_{n\geq -2q+1}$ be a solution of Eq. 1, with $A=-1$.

1. The following hold for each $r\in\{2q-1,2q-2,\ldots,q\}$.

(a) If $a_{-r}a_{-r+q}=2/B$, then the subsequences $(x_{2qn-r})_{n\geq 0}$ and $(x_{2qn-r+q})_{n\geq 0}$ are constants, indeed $x_{2qn-r}=a_{-r}$ and $x_{2qn-r+q}=a_{-r+q}$.
(b) If $a_{-r}a_{-r+q}\in(-\infty,min(0,2/B))\cup(max(0,2/B),\infty)$ then $(x_{2qn-r})_{n\geq 0}$ converges to zero and $(|x_{2qn-r+q}|)_{n\geq 0}$ diverges to infinity.
(c) If $a_{-r}a_{-r+q}\in(min(0,2/B),max(0,2/B))$ then $(x_{2qn-r+q})_{n\geq 0}$ converges to zero and $(|x_{2qn-r}|)_{n\geq 0}$ diverges to infinity.

2. The whole sequence $(x_n)_n$ converges if and only if $B>0$ and $a_{-2q+1}=a_{-2q+2}=\cdots=a_0=\pm\sqrt{2/B}$. In fact, in this case $(x_n)_{n\geq -2q+1}$ is constantly equal to $\pm\sqrt{2/B}$.

Theorem 5.3: (*The case* $A=1$) Let $(x_n)_{n\geq -2q+1}$ be a solution of Eq. 1. If $A=1$, then the sequence $(x_n)_{n\geq -2q+1}$ converges to zero.

Proof: Let $r\in\{2q-1,2q-2,\ \ldots\ ,q\}$. From Corollary (4.1), we can write x_{2qn-r} as $x_{2qn-r} = \frac{a_{-r}}{1+Ba_{-r}a_{-r+q}}\prod_{p=1}^{n-1}V_p$, where $V_p = 1-\frac{Ba_{-r}a_{-r+q}}{1+(2p+1)Ba_{-r}a_{-r+q}}$. There exists $N\in\mathbb{N}$ such that $V_p\in$

(0,1) for all $p > N$. So x_{2qn-r} can be written as $x_{2qn-r} = \frac{a_{-r}}{1+Ba_{-r}a_{-r+q}} exp\left(\sum_{p=N}^{n-1} lnV_p\right)$. The Taylor expansion of V_p implies that $ln\, V_p \sim -\frac{Ba_{-r}a_{-r+q}}{1+(2p+1)Ba_{-r}a_{-r+q}}$, which is the general term of divergent infinite series. Since $ln\, V_p < 0$, the series $\sum_{p\geq N} lnV_p$ diverges to $-\infty$, and hence the subsequence $\left(x_{2qn-r}\right)_{n\geq 0}$ converges to zero. Using a similar argument, we can show that for all $r \in \{q-1, q-2, \ldots, 0\}$, the subsequence $\left(x_{2qn-r}\right)_{n\geq 0}$ converges to zero. Therefore the whole sequence $(x_n)_{n\geq -2q+1}$ converges to zero.

Theorem 5.4: (*The case* $|A| > 1$) Let $(x_n)_{n\geq -2q+1}$ be a solution of Eq. 1. If $|A| > 1$, then for each $r \in \{2q-1, 2q-2, \ldots, 0\}$ the subsequence $(x_{2qn-r})_{n\geq 0}$ converges. Moreover, if $r \in \{2q-1, 2q-2, \ldots, q\}$, then one of the following statements is true.

1. If $A - 1 + Ba_{-r}a_{-r+q} = 0$, then the subsequences $\left(x_{2qn-r}\right)_{n\geq 0}$ and $\left(x_{2qn-r+q}\right)_{n\geq 0}$ are constants.
2. If $A - 1 + Ba_{-r}a_{-r+q} \neq 0$, then the subsequences $\left(x_{2qn-r}\right)_{n\geq 0}$ and $\left(x_{2qn-r+q}\right)_{n\geq 0}$ converge to zero.

Proof: Let $r \in \{2q-1, 2q-2, \ldots, q\}$. We will only show the convergence of the subsequences $\left(x_{2qn-r}\right)_{n\geq 0}$. The convergence of subsequence $\left(x_{2qn-r+q}\right)_{n\geq 0}$ can be established using a similar argument. If $A - 1 + Ba_{-r}a_{-r+q} = 0$, then the subsequence $\left(x_{2qn-r}\right)_{n\geq 0}$ is constant equal x_{-r}, and so it converges. Assume $A - 1 + Ba_{-r}a_{-r+q} \neq 0$ So we can write Eq. 14 as $x_{2qn-r} = \frac{a_{-r}}{(Ba_{-r}a_{-r+q}+A)A^{n-1}} \prod_{p=1}^{n-1} W_p$, where $W_p = \left(1 - \frac{1}{\alpha A^{2p}}\right) / \left(1 - \frac{1}{\alpha A^{2p+1}}\right)$ and $\alpha = \frac{A-1+Ba_{-r}a_{-r+q}}{Ba_{-r}a_{-r+q}}$. Since A^{1-n} converges to zero, it suffices to show that $\prod_{p\geq 1} W_p$ converges. Using the Taylor expansion of $W_p$we obtain that $W_p \sim 1 + \frac{1}{\alpha}\left(\frac{1}{A} - 1\right)\frac{1}{A^{2p}}$ Now there exists $N \in \mathbb{N}$ such that $W_p > 1$, for all $p \geq N$, or $0 < W_p < 1$, for all $p \geq N$. Since $\prod_{p\geq 1}\left(1 + \frac{1}{\alpha}\left(\frac{1}{A} - 1\right)\frac{1}{A^{2p}}\right)$ converges, by equivalence criterion $\prod_{p\geq 1} W_p$ converges. This complete the proof.

The following corollary is a direct consequence of the previous theorem.

Corollary 5.1: Let $(x_n)_{n\geq -2q+1}$be a solution of Eq. 1, with $|A| > 1$. Then the whole sequence $(x_n)_n$ converges if and only if one of the following conditions holds:

1. $\prod_{r=-2q+1}^{-q}\left(A - 1 + Ba_{-r}a_{-r+q}\right) \neq 0$,
2. $(1 - A)B > 0$ and $a_{-2q+1} = a_{-2q+2} = \cdots = a_0 = \pm\sqrt{(1 - A)/B}$.

6. Periodicity

We start this section by stating the following lemma that describes sufficient conditions in order for Eq. 1 to have a periodic solution.

Lemma 6.1: Let $(x_n)_{n\geq -2q+1}$ be a solution of Eq. 1. Suppose that for each $r \in \{2q-1, 2q-2, \ldots, 0\}$, $\left(x_{2qn-r}\right)_{n\geq 0}$ converges to a real number l_r. Let $(y_n)_{n\geq -2q+1}$ be the periodic$-2q$ solution whose initial $2q$ terms are given by $y_{-r} = l_r$, $r = 2q-1, 2q-2, \ldots, 0$. Then the sequence $(y_n)_{n\geq -2q+1}$ is a periodic$-2q$ solution of Eq. 1.

We call $(y_n)_{n\geq -2q+1}$ the periodic$-2q$ solution induced by the $(x_n)_{n\geq -2q+1}$. Clearly, every periodic$-2q$ solution induces itself. The periodicity results are given in the following Theorem.

Theorem 6.1:

1. If $|A| < 1$, then every solution of Eq. 1 induces a periodic$-2q$ solution.
2. If $A = -1$, then a non-trivial solution $(x_n)_{n\geq -2q+1}$ of Eq. 1 is periodic$-2q$ if and only if its initial data satisfy $a_{-r}a_{-r+q} = 2/B$ for all $r \in \{2q-1, \ldots, q\}$.
3. If $A = 1$, then Eq. 1 has no non-trivial periodic$-2q$ solution.
4. If $|A| > 1$, then every solution of Eq. 1 induces a periodic$-2q$ solution. Moreover, a non-trivial solution $(x_n)_{n\geq -2q+1}$ is itself a periodic$-2q$ solution if and only if $a_{-r}a_{-r+q} = (1 - A)/B$ for all $r \in \{2q-1, \ldots, q\}$.

Proof: Recall first that a periodic$-p$ sequence is completely determined by giving p successive terms.

1. Suppose that $|A| < 1$, and let $(x_n)_{n\geq -2q+1}$ be a solution of Eq. 1. Then, by Theorem (5.1), for each $r \in \{2q-1, 2q-2, \ldots, 0\}$, the subsequence $\left(x_{2qn-r}\right)_{n\geq 0}$converges to a real number l_r. So by Lemma (6.1), $(x_n)_{n\geq -2q+1}$ induces the periodic$-2q$ solution $l_{2q-1}, l_{2q-2}, \ldots, l_0, l_{2q-1}, l_{2q-2}, \ldots, l_0, \ldots$
2. Suppose $A = -1$, and let $(x_n)_{n\geq -2q+1}$ be a non-trivial solution of Eq. 1. Then Theorem (5.2) and Remark (5.2) imply that for each $r \in \{2q-1, 2q-2, \ldots, q\}$, if $a_{-r}a_{-r+q} = 2/B$ then $\left(x_{2qn-r}\right)_{n\geq 0}$ is constant. Therefore $(x_n)_{n\geq -2q+1}$ is a periodic$-2q$ solution. On the other hand, if $a_{-r}a_{-r+q} \neq 2/B$ for some $r \in \{2q-1, 2q-2, \ldots, q\}$, then either $\left(x_{2qn-r}\right)_{n\geq 0}$or $\left(x_{2qn-r+q}\right)_{n\geq 0}$ diverges, and hence $(x_n)_{n\geq -2q+1}$ is not a periodic$-2q$ solution.
3. Let $(x_n)_{n\geq -2q+1}$ be a non-trivial solution of Eq. 1 with $A = 1$. According to Theorem (5.3), the sequence $(x_n)_{n\geq -2q+1}$ converges to zero. By Lemma (6.1), it follows that the zero sequence is the only periodic solution of Eq. 1.
4. Let $|A| > 1$, and let $(x_n)_{n\geq -2q+1}$ be a non-trivial solution of Eq. 1. So Theorem (5.4) states that for

each $r \in \{2q-1,\ 2q-2, \ldots,\ 0\}$, the subsequence $(x_{2qn-r})_{n\geq 0}$ converges, and so $(x_n)_{n\geq -2q+1}$ induces a periodic$-2q$ solution of Eq. 1. In addition, $(x_n)_{n\geq -2q+1}$ is periodic if and only if $(x_{2qn-r})_{n\geq 0}$ is constant for all $r \in \{2q-1,\ 2q-2, \ldots,\ 0\}$. Hence the result.

7. Numerical simulations

$$x_{n+1} = \frac{x_{n-2q+1}}{A + Bx_{n-2q+1}x_{n-q+1}}$$

where A and B are arbitrary constants and with arbitrary initial data $x_{-2q+1} = a_{-2q+1}$, $x_{-2q+2} = a_{-2q+2}, \ldots, x_0 = a_0, q = 1, 2, \ldots$

1. The case $|A| < 1$, $q = 1$ is studied using the parameter values $A = 1/2$, $B = 3$ and the initial data $a_{-1} = -2$, $a_0 = 1$. In Fig. 1, it is shown that the subsequences $(x_{2n-1})_n$ and $(x_{2n})_n$ converge which match Theorem (5.1). The solution is bounded.

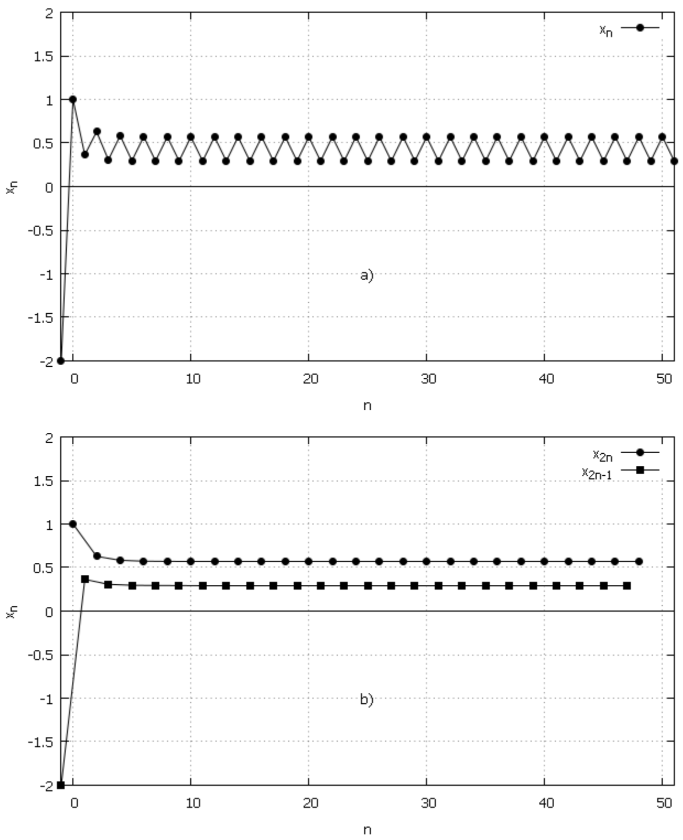
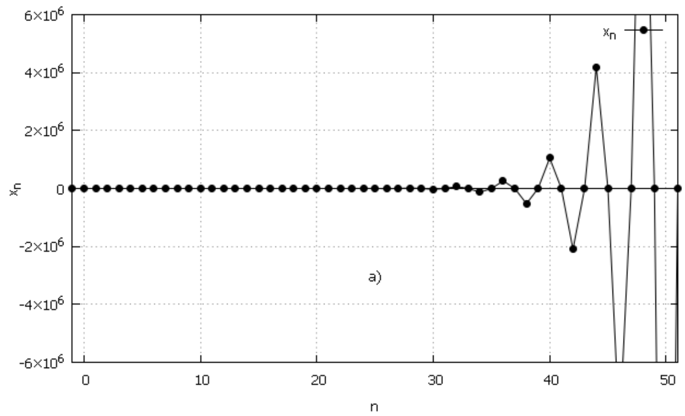


Fig. 1: (a) The whole sequence diverges (b) The corresponding subsequences converges

2. The case $A = -1$, $q = 1$, $a_{-r}a_{-r+q} \in (-\infty, min(0,2/b)) \cup (max(0,2/b), \infty)$ is investigated using the parameter values $B = 1/3$ and the initial data $a_{-1} = -3$, $a_0 = 1$. In Fig. 2, we notice that the subsequence $(x_{2n-1})_n$ converges to zero, however the subsequence $(|x_{2n}|)_n$ goes to infinity. This is justified analytically in the proof of Theorem (5.2) and Remark (5.2). The whole solution is unbounded.



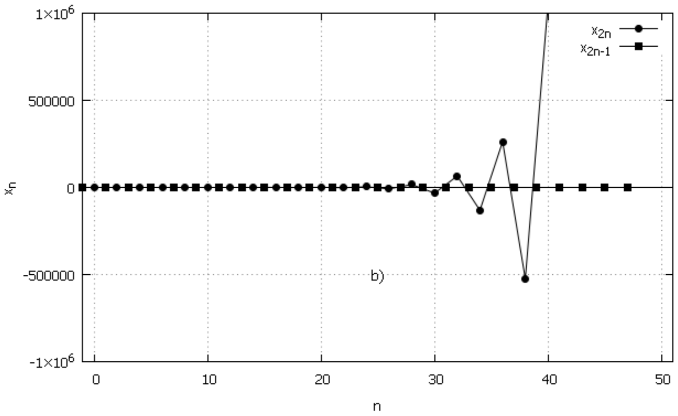


Fig. 2: (a) The whole sequence diverges (b) The corresponding subsequences are illustrated

3. The case $A=1$, $q=2$ is studied using the parameter values $B=3$ and the initial data $a_{-3}=1/2$, $a_{-2}=3$, $a_{-1}=-2$, $a_0=1$. In Fig. 3, it is clear that the solution is dumping to zero. This is justified analytically in the proof of Theorem (5.3).

4. The case $|A|>1$, $q=2$, $A-1+Ba_{-r}a_{-r+q}=0$ and $A-1+Ba_{-r}a_{-r+q}\neq 0$ can be obtained by choosing the parameter values $A=5$, $B=2$ and the initial data $a_{-3}=1$, $a_{-2}=3$, $a_{-1}=2$, $a_0=2$. Since $A-1+Ba_{-3}a_{-1}\neq 0$, the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ converge to zero, and since $A-1+Ba_{-2}a_0\neq 0$ the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero (Fig. 4). These observations are coherent to the Theorem (5.4).

5. The case $|A|<1$, $q=3$ is discussed using the parameter values A=1/2, $B=3$ and the initial data $a_{-5}=1/4$, $a_{-4}=2$, $a_{-3}=1/2$, $a_{-2}=-3$, $a_{-1}=1$, $a_0=-2$. As shown in Fig. 5, the simulation results confirm that the subsequences $(x_{6n-r})_n$, $r=0,1,\ldots,5$ converge, which match Theorem (5.1).

6. The case $A=-1$, $q=3$, $a_{-r}a_{-r+q}\in\left(-\infty,min(0,2/\mathrm{B})\right)\cup(max(0,2/\mathrm{B}),\infty)$, $a_{-r}a_{-r-q}\in\left(min(0,2/\mathrm{B}),max(0,2/\mathrm{B})\right)$ is illustrated in Fig. 6 in which we set the parameter values $B=1/2$ and the initial data $a_{-5}=-2$, $a_{-4}=3$, $a_{-3}=2$, $a_{-2}=1$, $a_{-1}=1$, $a_0=1/2$. Since $a_{-2}a_{-5}\in(-\infty,0)\cup(2/B,\infty)$, it follows that $(x_{6n-5})_n$ converges to zero and $(|x_{6n-2}|)_n$ goes to infinity, and since $a_{-1}a_{-4}\in(0,2/B)$, $(|x_{6n-4}|)_n$ goes to infinity and $(x_{6n-1})_n$ converges to zero. Also because $a_0a_{-3}\in(-\infty,0)\cup(2/B,\infty)$, we get $(x_{6n-3})_n$ converges to zero and $(|x_{6n}|)_n$ goes to infinity. This is justified analytically in the proof of Theorem (5.2) and Remark (5.2).

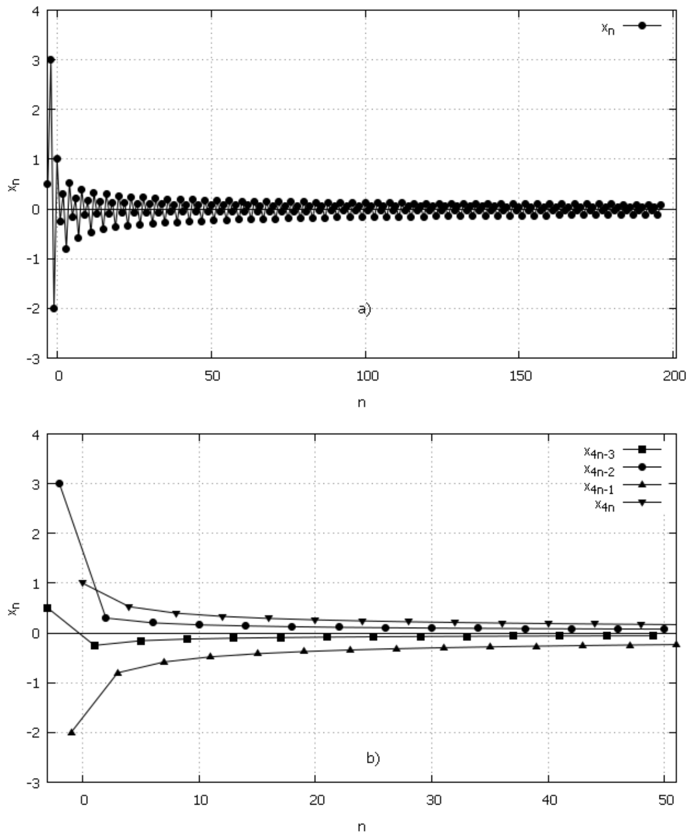


Fig. 3: (a) The whole sequence converges to zero (b) The corresponding subsequences converges to zero

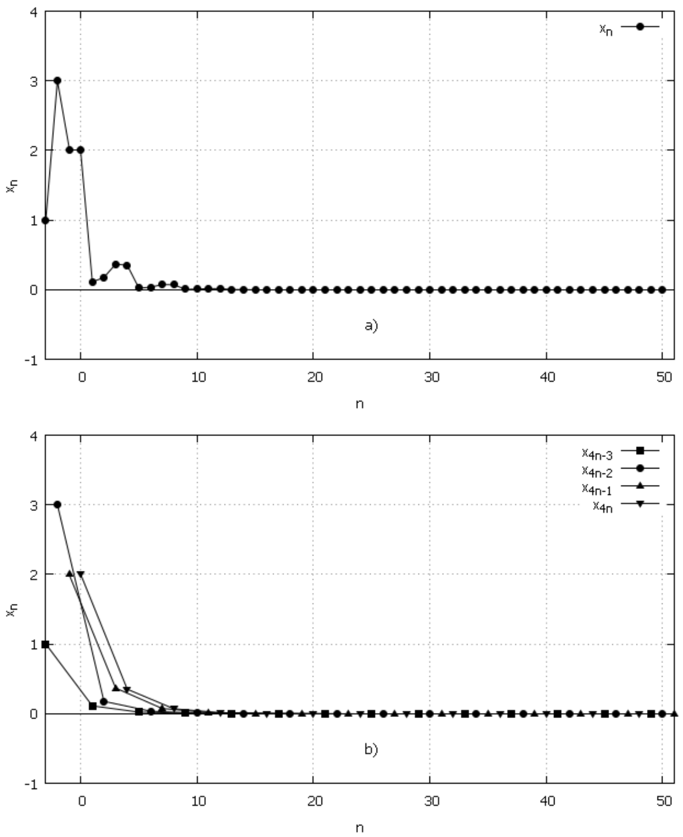


Fig. 4: (a) The whole sequence converges to zero (b) The corresponding subsequences converge to zero

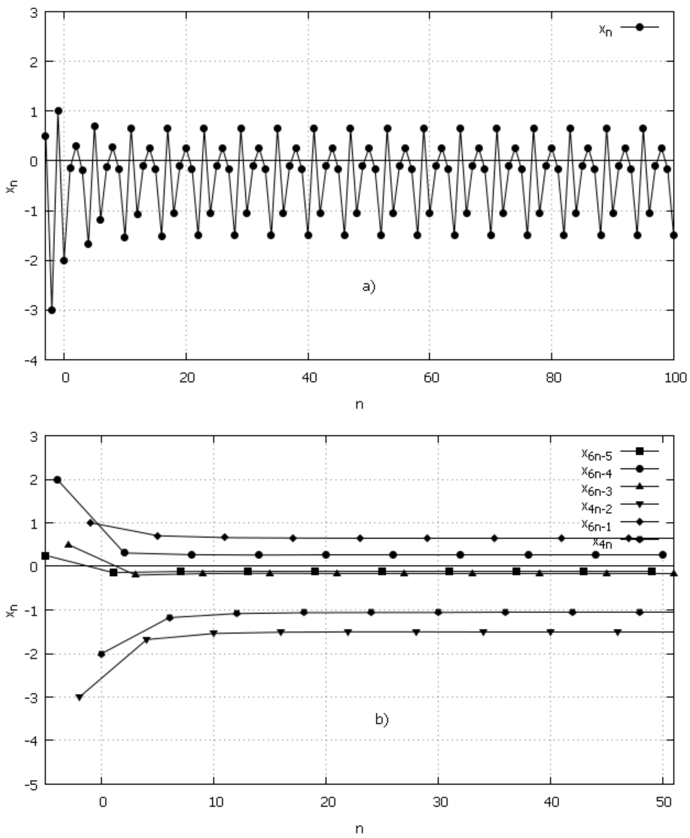


Fig. 5: (a) The whole sequence diverges (b) The corresponding subsequences converge

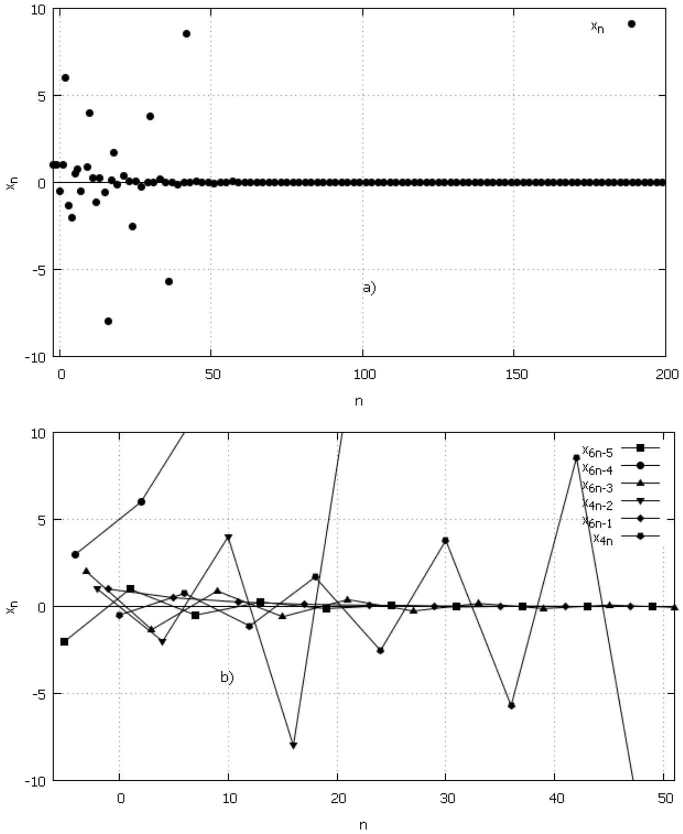


Fig. 6: (a) The whole sequence diverges and unbounded (b) The corresponding subsequences are illustrated

7. The case $A = -1$, $q = 3$ is investigated using the parameter values $B = 1/2$ and the initial data $a_{-5} = 4$, $a_{-4} = -3$ $a_{-3} = -1$, $a_{-2} = 1$, $a_{-1} = -4/3$, $a_0 = 2$. We see that $a_{-2}a_{-5} = a_{-1}a_{-4} = 2/B$, so $(x_{6n-5})_n$, $(x_{6n-2})_n$, $(x_{6n-4})_n$, $(x_{6n-1})_n$ are constants (Fig. 7). However, $a_0a_{-3} \in (-\infty, 0) \cup (2/B, \infty)$. So $(x_{6n-3})_n$ converges to zero while $(|x_{6n}|)_n$ goes to infinity which matches Theorem (5.2) and Remark (5.2).

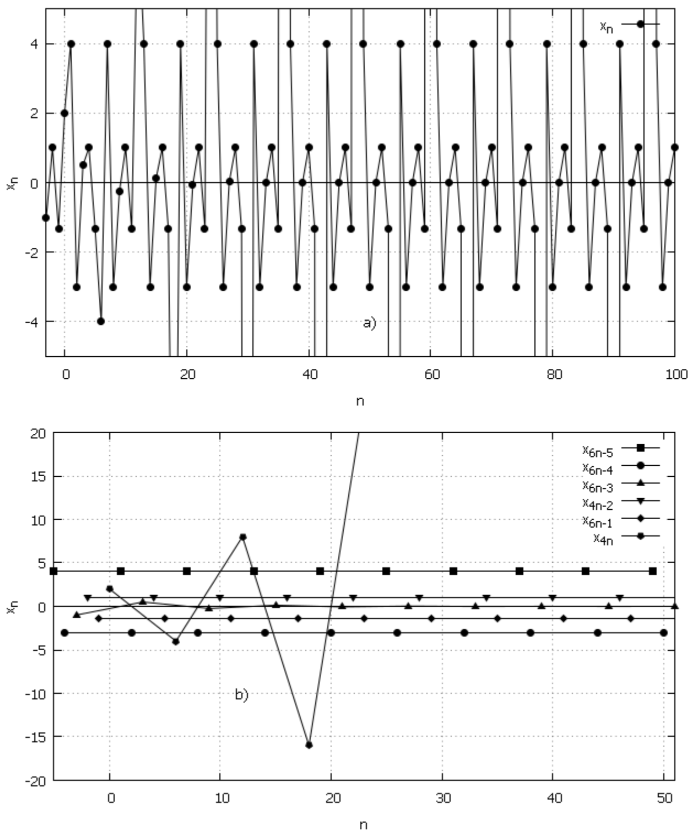


Fig. 7: (a) The whole sequence diverges and unbounded (b) The corresponding subsequences are illustrated

8. The case $A = 1$, $q = 3$ is obtained by choosing the parameter values $B = 2$ and the initial data $a_{-5} = -2$, $a_{-4} = 1$, $a_{-3} = 2$, $a_{-2} = 0.2$, $a_{-1} = 0.3$, $a_0 = 0.4$ (Fig. 8). The whole sequence $(x_n)_n$ converges to zero. This is justified analytically in the proof of Theorem (5.3).

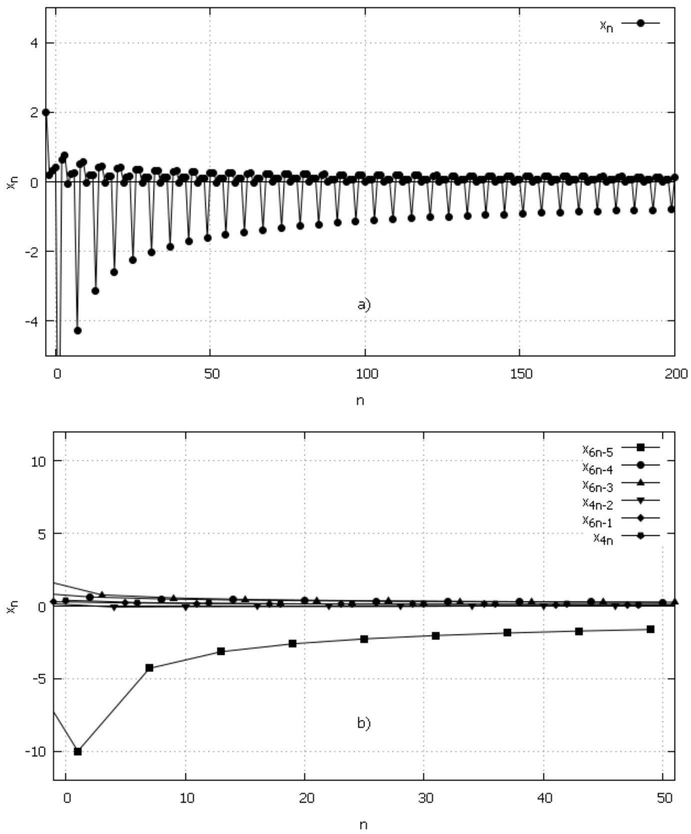


Fig. 8: (a) The whole sequence $(x_n)_n$ converges to zero (b) The corresponding subsequences converge to zero

8. Conclusion

In this paper, we have presented a complete study of the class of rational difference equations $x_{n+1} = \frac{x_{n-2q+1}}{A+Bx_{n-2q+1}x_{n-q+1}}$ with arbitrary initial data, where A and B are arbitrary parameters, and q is an arbitrary nonnegative integer. Keeping the order 2q as an arbitrary parameter allowed us to make a significant contribution and improves and surpasses studies of several forms of difference equations that have been investigated in the literature. In this study, we have given a detailed analytical investigation of the convergence of the solutions including their dependence on parameters and initial data. We also provided a complete discussion of the local and global stability of the equilibrium points as well as the existence of periodic solutions of this class. At the end, numerical simulations have been done to confirm the correctness of analytical results.

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Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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