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A note on degenerate poly-Genocchi polynomials

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ABSTRACT

In this article, we introduce degenerate poly-Genocchi polynomials and numbers. We derive summation formulas, recurrence relations, and identities of these polynomials by using summation techniques series. Also, we establish symmetric identities by using power series methods, respectively.

1. Introduction

Genocchi polynomials are very frequently used in various problems in pure and applied mathematics related to functional equations, number theory, complex analytic number theory, Homotopy theory (stable Homotopy groups of spheres), differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic number theory (p-adic L-functions), quantum physics (quantum Groups). For instance, generating functions for Genocchi polynomials with their congruence properties, recurrence relations, computational formulae and symmetric sum involving these polynomials have been studied by many authors in recent years such as Young (2008), Araci (2014), Araci et al. (2011), Açıkgöz et al. (2011), Araci et al. (2014a, 2014b), Haroon and Khan (2018), Khan et al. (2017, 2018), Khan and Haroon (2016), and Araci (2012).

The well-known degenerate exponential function (Kim et al., 2019; Kim and Ryoo, 2018) is defined by

$$e_\mu^\eta(z) = (1+\mu z)^{\frac{\eta}{\mu}}, e_\mu(z) = e_\mu^1(z), (\mu \in \mathbb{R}). \quad (1.1)$$

Since

$$\lim_{\mu\to 0}(1+\mu z)^{\frac{\eta}{\mu}} = e^{\eta z}.$$

The degenerate type Bernoulli and Euler polynomials are defined by (Carlitz, 1979; Carlitz, 1956):

$$\frac{z}{e_\mu(z)-1}e_\mu^\eta(z) = \frac{z}{(1+\mu z)^{\frac{1}{\mu}}-1}(1+\mu z)^{\frac{\eta}{\mu}} = \sum_{s=0}^{\infty} \beta_s(\eta;\mu)\frac{z^s}{s!}, \quad (1.2)$$

and

$$\frac{2}{e_\mu(z)+1}e_\mu^\eta(z) = \frac{2}{(1+\mu z)^{\frac{1}{\mu}}-1}(1+\mu z)^{\frac{\eta}{\mu}} = \sum_{s=0}^{\infty} \mathfrak{E}_s(\eta;\mu)\frac{z^s}{s!}. \quad (1.3)$$

Thus, we have

$$\lim_{\mu\to 0}\beta_s(x;\mu) = B_s(\eta), \lim_{\mu\to 0}\mathfrak{E}_s(\eta;\lambda) = E_s(\eta).$$

In the year 2016, Lim (2016) introduced the generalized degenerate type Genocchi polynomials $G_j^{(p)}(\eta;\mu)$ are defined by

$$\left(\frac{2z}{e_\mu(z)+1}\right)^p e_\mu^\eta(z) = \left(\frac{2z}{(1+\mu z)^{\frac{1}{\mu}}-1}\right)^p (1+\mu z)^{\frac{\eta}{\mu}} = \sum_{j=0}^{\infty} G_j^{(p)}(\eta;\mu)\frac{z^j}{j!}, \quad (1.4)$$

so that

$$G_j^{(p)}(\eta;\mu) = \sum_{s=0}^{j}\binom{j}{s}G_s^{(p)}(\mu)\left(\frac{\eta}{\mu}\right)_{j-s}. \quad (1.5)$$

From Eq. 1.4, we note that

$$\lim_{\mu\to 0}\sum_{s=0}^{\infty} G_j^{(p)}(\eta;\mu)\frac{z^j}{j!} = \lim_{\mu\to 0}\left(\frac{2z}{(1+\mu z)^{\frac{1}{\mu}}-1}\right)^p (1+\mu z)^{\frac{\eta}{\mu}}$$
$$= \left(\frac{2z}{e^z+1}\right)^p e^{\eta z} = \sum_{j=0}^{\infty} G_j^{(p)}(\eta)\frac{z^j}{j!}, (\text{see}(1-25)).$$

The degenerate poly-Bernoulli and poly-Genocchi polynomials are defined by (Khan, 2016a; Kim et al.,

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2019; Kim and Kim, 2015; Kim et al., 2014a; 2014b; Kim and Ryoo, 2018; Lim, 2016):

$$\frac{\mathrm{Li}_k(1-e^{-z})}{e_\mu(z)-1} e_\mu^\eta(z) = \frac{\mathrm{Li}_k(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}-1}(1+\mu z)^{\frac{\eta}{\mu}} = \sum_{s=0}^{\infty} B_s^{(k)}(\eta;\mu)\frac{z^s}{s!}, \quad (1.6)$$

the classical polylogarithm function $\mathrm{Li}_\nu(z)$ is

$$\mathrm{Li}_\nu(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^k}, (\nu \in \mathbb{Z})(\text{see}[18-24]) \quad (1.7)$$

so for $\nu \le 1$,

$$\mathrm{Li}_\nu(z) = -\ln(1-z), \mathrm{Li}_0(z) = \frac{z}{1-z}, \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \ldots$$

Kim et al. (2015, 2014a, 2014b) and Khan (2016b, 2016c) introduced polytype (Bernoulli and Genocchi) polynomials are defined by

$$\frac{\mathrm{Li}_\nu(1-e^{-z})}{e^z-1} e^{\xi z t} = \sum_{j=0}^{\infty} B_j^{(\nu)}(\xi)\frac{z^j}{j!}, \quad (1.8)$$

and

$$\frac{2\mathrm{Li}_\nu(1-e^{-z})e^{\xi z}}{e^z+1} = \sum_{j=0}^{\infty} G_j^{(\nu)}(\xi)\frac{z^j}{j!}. (\nu \in \mathbb{Z}). \quad (1.9)$$

For $\nu = 1$ in Eq. 1.8) and Eq. 1.9, we get

$$\frac{\mathrm{Li}_1(1-e^{-z})}{e^z-1} e^{\xi z} = \frac{z}{e^z-1} e^{\xi z} = \sum_{j=0}^{\infty} B_j(\xi)\frac{z^j}{j!}, \quad (1.10)$$

and

$$\frac{2\mathrm{Li}_1(1-e^{-z})}{e^z+1} e^{\xi z} = \frac{2z}{e^z+1} e^{\xi z} = \sum_{j=0}^{\infty} G_j(\xi)\frac{z^j}{j!}. \quad (1.11)$$

From Eq. 1.10 and Eq. 1.11, we have

$$B_j^{(1)}(\xi) = B_j(\xi), G_j^{(1)}(\xi) = G_j(\xi).$$

We recall the following definition as follows:

- The first kind of Stirling numbers are given by

$$(a)_j = a(a-1)\cdots(a-j+1) = \sum_{q=0}^{j} S_1(j,q)a^q, (j \ge 0). \quad (1.12)$$

- The second kind of Stirling numbers are defined by generating function

$$(e^z-1)^p = p!\sum_{q=p}^{\infty} S_2(q,p)\frac{z^q}{q!}. \quad (1.13)$$

A generalized falling factorial sum $\tau_k(j;\mu)$ is defined by Young (2008):

$$\sum_{k=0}^{\infty} \tau_k(j;\mu)\frac{z^k}{k!} = \frac{1-(-(1+\mu z))^{\frac{(n+1)}{\mu}}}{1+(1+\mu z)^{\frac{1}{\mu}}}, \quad (1.14)$$

where $\lim_{\mu\to 0}\tau_k(j;\mu) = T_k(j)$. This article is as follows. We consider the degenerate type poly-Genocchi polynomials $G_j^{(\nu)}(\xi;\mu)$ and construct some basic properties and derive some implicit formulae and symmetric identities for the degenerate poly-Genocchi polynomials by using different analytical means of their respective generating functions.

2. Degenerate poly-Genocchi polynomials

Let $\mu \in \mathbb{C}, \nu \in \mathbb{Z}$. We introduce the degenerate poly-Genocchi polynomials and numbers are given by

$$\frac{2Li_\nu(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi}{\mu}} = \sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!}, \quad (2.1)$$

so that

$$G_j^{(\nu)}(\xi;\mu) = \sum_{k=0}^{j} \binom{j}{k} G_{k,\mu}^{(\nu)}(\xi)\left(-\frac{\xi}{\mu}\right)_{j-k}(-\mu)^{j-k}. \quad (2.2)$$

When $\xi = 0$ in Eq. 2.1, $G_{j,\mu}^{(\nu)} = G_{j,\mu}^{(\nu)}(0)$. Note that

$$G_{j,\mu}^{(1)}(\xi) = G_{j,\mu}(\xi),$$

and

$$\lim_{\mu\to 0} G_j^{(\nu)}(\xi;\mu) = G_j^{(\nu)}(\xi), (\nu \in \mathbb{Z}), \quad (2.3)$$

where $G_j^{(\nu)}(\xi)$ are called the poly-Genocchi polynomials.

Theorem 2.1: Let $\mu \in \mathbb{C}, \nu \in \mathbb{Z}$, and $j \ge 0$. Then

$$G_j^{(2)}(\xi;\mu) = \sum_{s=0}^{j} \binom{j}{s}\frac{B_s s!}{s+1} E_{j-s}(\xi;\mu). \quad (2.4)$$

Proof: Using Eq. 2.1, we see

$$\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!} = \frac{2Li_\nu(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi}{\mu}}$$

$$= \frac{2(1+\mu z)^{\frac{\xi}{\mu}}}{(1+\mu z)^{\frac{1}{\mu}}+1} \underbrace{\int_0^w \frac{1}{e^v-1}\int_0^v \frac{1}{e^v-1}\cdots\frac{1}{e^v-1}\int_0^v \frac{v}{e^v-1}}_{(\nu-2)-\ times} dv\cdots dv \quad (2.5)$$

for $\nu = 2$ in Eq. 2.5, we find

$$\sum_{j=0}^{\infty} G_j^{(2)}(\xi;\mu)\frac{z^j}{j!} = \frac{2(1+\mu z)^{\frac{\xi}{\mu}}}{(1+\mu z)^{\frac{1}{\mu}}+1}\int_0^w \frac{u}{e^u-1}du$$

$$= \left(\sum_{s=0}^{\infty} \frac{B_s z^s}{s+1}\right)\frac{2(1+\mu z)^{\frac{\xi}{\mu}}}{(1+\mu z)^{\frac{1}{\mu}}+1}$$

$$= \left(\sum_{s=0}^{\infty} \frac{B_s s!}{(s+1)s!}\frac{z^s}{s!}\right)\left(\sum_{j=0}^{\infty} E_j(\xi;\mu)\frac{z^j}{j!}\right)$$

$$= \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\binom{j}{s}\frac{B_s s!}{m+1}E_{j-s}(\xi;\mu)\right)\frac{z^j}{j!},$$

which gives the asserted result of Eq. 2.4.

Theorem 2.2: Let $\mu \in \mathbb{C}, \nu \in \mathbb{Z}$, and $j \ge 0$. Then

$$G_j^{(\nu)}(\xi;\mu) = \sum_{s=0}^{j}\binom{j}{s} \left(\sum_{q=1}^{s+1}\frac{(-1)^{q+s+1}q!S_2(s+1,q)}{q^k(s+1)}\right) G_{j-s}^{(\nu)}(\xi;\mu). \quad (2.6)$$

Proof: Eq. 2.1, we find

$$\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!} = \left(\frac{\mathrm{Li}_\nu(1-e^{-z})}{z}\right)\left(\frac{2z(1+\mu z)^{\frac{\xi}{\mu}}}{(1+\mu z)^{\frac{1}{\mu}}+1}\right) \quad (2.7)$$

now

$$\frac{\mathrm{Li}_\nu(1-e^{-z})}{z} = \frac{1}{z}\sum_{q=1}^{\infty}\frac{(1-e^{-z})^q}{q^k} = \frac{1}{z}\sum_{q=1}^{\infty}\frac{(-1)^q}{l^k}(1-e^{-z})^q$$
$$= \frac{1}{z}\sum_{q=1}^{\infty}\frac{(-1)^q}{q^k}q!\sum_{s=l}^{\infty}(-1)^s S_2(s,q)\frac{z^s}{s!}$$
$$= \frac{1}{s}\sum_{s=1}^{\infty}\sum_{q=1}^{s}\frac{(-1)^{q+s}}{q^k}q!\,S_2(s,q)\frac{a^s}{s!}$$
$$= \sum_{s=0}^{\infty}\left(\sum_{q=1}^{s+1}\frac{(-1)^{q+s+1}}{q^k}q!\frac{S_2(s+1,q)}{s+1}\right)\frac{z^s}{s!}. \quad (2.8)$$

From Eq. 2.7 and Eq. 2.8, we have

$$\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!} =$$
$$\sum_{s=0}^{\infty}\left(\sum_{q=1}^{s+1}\frac{(-1)^{q+s+1}}{q^k}q!\frac{S_2(s+1,q)}{s+1}\right)\frac{z^s}{s!}\left(\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!}\right),$$

which proves the result of Eq. 2.6.

Theorem 2.3: Let $\mu \in \mathbb{C}, \nu \in \mathbb{Z}$, and $j \geq 0$. Then

$$\frac{1}{2}\left[G_j^{(\nu)}(\xi+1;\mu) + G_j^{(\nu)}(\xi;\mu)\right]$$
$$= \sum_{s=1}^{j}\binom{j}{s}\left(\sum_{q=0}^{s-1}\frac{(-1)^{q+s+1}}{(q+1)^k}(q+1)!\,S_2(s,q+1)\right)\left(-\frac{\xi}{\mu}\right)_{j-s}(-\mu)^{j-s}. \quad (2.9)$$

Proof: Using the definition in Eq. 2.1, we have

$$\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi+1;\mu)\frac{z^j}{j!} + \sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!}$$
$$= \frac{2Li_\nu(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi+1}{\mu}} + \frac{2Li_\nu(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi}{\mu}}$$
$$= 2Li_\nu(1-e^{-z})(1+\mu z)^{\frac{\xi}{\mu}}$$
$$= 2\sum_{q=0}^{\infty}\frac{(1-e^{-z})^{q+1}}{(q+1)^k}(1+\mu z)^{\frac{\xi}{\mu}}$$
$$= 2\sum_{s=1}^{\infty}\left(\sum_{q=0}^{s-1}\frac{(-1)^{q+s+1}}{(q+1)^k}(s+1)!\,S_2(s,q+1)\right)\frac{z^s}{s!}(1+\mu z)^{\frac{\xi}{\mu}}$$
$$= 2\left(\sum_{s=1}^{\infty}\left(\sum_{q=0}^{s-1}\frac{(-1)^{q+s+1}}{(q+1)^k}(q+1)!\,S_2(s,q+1)\right)\frac{z^s}{s!}\right)\left(\sum_{j=0}^{\infty}\left(-\frac{\xi}{\mu}\right)_j(-\mu)^j\frac{z^j}{j!}\right),$$

yields the result in Eq. 2.9.

Theorem 2.4: Let $c \in \mathbb{N}, \nu \in \mathbb{Z}$, and $j \geq 0$. Then

$$G_j^{(\nu)}(\xi;\mu)$$
$$= \sum_{a=0}^{d-1}\sum_{v=0}^{j}\sum_{v=1}^{v+1}\binom{j}{v}c^{j-v-1}\frac{(-1)^{v+s+1}s!S_2(v+1,s)}{s^k v+1}$$
$$G_{j-v}^{(\nu)}\left(\frac{v+\xi}{c};\frac{\mu}{c}\right). \quad (2.10)$$

Proof: Using Eq. 2.1, we find

$$\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!} = \frac{2Li_\nu(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi+1}{\mu}}$$
$$= \frac{2Li_\nu(1-e^{-z})}{(1+\mu z)^{\frac{c}{\mu}}+1}\sum_{a=0}^{c-1}(1+\mu z)^{\frac{l+\xi}{\mu}}$$
$$= \left(\frac{\mathrm{Li}_\nu(1-e^{-z})}{z}\right)\frac{1}{c}\sum_{a=0}^{c-1}\frac{2cz}{(1+\mu z)^{\frac{c}{\mu}}+1}(1+\mu z)^{\frac{l+\xi}{\mu}}$$
$$= \left(\sum_{v=0}^{\infty}\left(\sum_{s=1}^{v+1}\frac{(-1)^{v+s+1}}{s^k}s!\frac{S_2(v+1,s)}{v+1}\right)\frac{z^v}{v!}\right)$$
$$\left(\sum_{j=0}^{\infty}c^{j-1}\sum_{a=0}^{c-1}G_j\left(\frac{l+\xi}{c};\frac{\mu}{c}\right)\frac{z^j}{j!}\right),$$

which gives the result in Eq. 2.10.

Theorem 2.5: Let $\nu \in \mathbb{Z}$ and $n \geq 0$, then

$$G_j^{(\nu)}(\xi+u;\mu) = \sum_{r=0}^{j}\binom{j}{r}G_{j-r}^{(\nu)}(\mu)^r\left(\frac{u}{\mu}\right)_r. \quad (2.11)$$

Proof: Using Eq. 2.1, we see

$$\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!} = \frac{2Li_\nu(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi+1}{\mu}}$$
$$= \left(\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!}\right)\left(\sum_{r=0}^{\infty}(\mu)^r\left(\frac{u}{\mu}\right)_r\frac{z^r}{r!}\right).$$

On comparing the coefficients of $\frac{z^j}{j!}$, we get the result in Eq. 2.11.

Theorem 2.6: Let $\nu \in \mathbb{Z}$ and $n \geq 0$. Then

$$G_j^{(\nu)}(\xi;\mu) = \sum_{r=0}^{j}\binom{j}{r}\left(-\frac{w}{\mu}\right)_r(-\mu)^r G_{j-r}^{(\nu)}(\xi-w;\mu). \quad (2.12)$$

Proof: Using Eq. 2.1, we find

$$\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!} = \frac{2Li_\nu(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi-w}{\mu}}(1+\mu z)^{\frac{w}{\mu}}$$
$$= \left(\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!}\right)\left(\sum_{r=0}^{\infty}\left(-\frac{w}{\mu}\right)_r\frac{(-\mu z)^r}{r!}\right),$$

which completes the result in Eq. 2.12.

Theorem 2.7: Let $\nu \in \mathbb{Z}$ and $n \geq 0$, then

$$G_j^{(\nu)}(\xi+1;\mu)\frac{z^j}{j!} = \sum_{r=0}^{j}G_{j-r}^{(\nu)}(\xi;\mu)\left(-\frac{1}{\mu}\right)_r(-\mu)^r\frac{z^j}{(j-r)!r!}. \quad (2.13)$$

Proof: Using Eq. 2.1, we see

$$\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi+1;\mu)\frac{z^j}{j!} - \sum_{n=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!} =$$
$$\frac{2\mathrm{Li}_\nu(1-e^{-z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi}{\mu}}((1+\mu z)^{\frac{1}{\mu}}-1)$$
$$= \left(\sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!}\right)\left(\sum_{r=0}^{\infty}\left(-\frac{1}{\mu}\right)_r\frac{(-\mu z)^r}{r!}\right) - \sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!}$$
$$= \sum_{j=0}^{\infty}\sum_{r=0}^{j}G_{j-r}^{(\nu)}(\xi;\mu)\left(-\frac{1}{\mu}\right)_r(-\mu)^r\frac{z^j}{(j-r)!r!} - \sum_{j=0}^{\infty} G_j^{(\nu)}(\xi;\mu)\frac{z^j}{j!},$$

which gives the result in Eq. 2.13.

3. General identities

Here, we establish general identities for the degenerate poly-Genocchi polynomials $G_j^{(\nu)}(\xi;\mu)$ by applying the generating function. We start the some identities as follows.

Theorem 3.1: Let $\nu \in \mathbb{Z}, a, b > 0$, and $n \geq 0$. Then

$$\sum_{r=0}^{j}\binom{j}{r}b^r a^{j-r}G_{j-r}^{(\nu)}(b\xi;\mu)G_r^{(\nu)}(a\xi;\mu)$$
$$= \sum_{r=0}^{j}\binom{j}{r}a^r b^{j-r}G_{j-r}^{(\nu)}(a\xi;\mu)G_r^{(r)}(b\xi;\mu). \quad (3.1)$$

Proof: Suppose

$$A(z)=\left(\frac{4\mathrm{Li}_\nu(1-e^{-az})\mathrm{Li}_\nu(1-e^{-bz})}{((1+\mu z)^{\frac{a}{\mu}}+1)((1+\mu z)^{\frac{b}{\mu}}+1)}\right)(1+\mu z)^{\frac{2ab\xi}{\mu}}. \quad (3.2)$$

$$A(z)=\sum_{j=0}^{\infty}G_j^{(\nu)}(b\xi;\mu)\frac{(az)^j}{j!}\sum_{r=0}^{\infty}G_r^{(\nu)}(a\xi;\mu)\frac{(bz)^r}{r!}$$
$$=\sum_{j=0}^{\infty}\sum_{r=0}^{j}\binom{j}{r}a^{j-r}b^rG_{j-r}^{(\nu)}(b\xi;\mu)G_r^{(\nu)}(a\xi;\mu)\frac{z^j}{j!}.$$

Similarly, we can show that

$$A(z)=\sum_{j=0}^{\infty}G_{j,\mu}^{(\nu)}(a\xi;\mu)\frac{(bz)^j}{j!}\sum_{r=0}^{\infty}G_{r,}^{(\nu)}(b\xi;\mu)\frac{(az)^r}{r!}$$
$$=\sum_{j=0}^{\infty}\sum_{r=0}^{j}\binom{j}{r}a^rb^{j-r}G_{j-r}^{(\nu)}(a\xi;\mu)G_r^{(\nu)}(b\xi;\mu)\frac{z^j}{j!},$$

which proves the identity in Eq. 3.1.

Remark 3.1: Letting $b=1$, Theorem 4.1reduces the following result

$$\sum_{r=0}^{j}\binom{j}{r}a^{j-r}G_{j-r}^{(\nu)}(\xi;\mu)G_r^{(\nu)}(a\xi;\mu)$$
$$=\sum_{r=0}^{j}\binom{j}{r}a^rG_{j-r}^{(\nu)}(a\xi;\mu)G_r^{(r)}(\xi;\mu). \quad (3.3)$$

Theorem 3.2: Let $\nu\in\mathbb{Z}$, $a,b>0$, and $n\geq 0$. Then

$$\sum_{r=0}^{j}\binom{j}{r}a^{j-r}b^rG_{j-r}^{(\nu)}(b\xi;\mu)\sum_{i=0}^{r}\binom{r}{i}\tau_i(a-1;\mu)G_{r-i}^{(\nu)}(a\eta;\mu)$$
$$=\sum_{r=0}^{j}\binom{j}{r}a^rb^{j-r}G_{j-r}^{(\nu)}(a\xi;\mu)\sum_{i=0}^{r}\binom{r}{i}\tau_i(b-1;\mu)G_{r-i}^{(\nu)}(b\eta;,\mu). \quad (3.4)$$

Proof: We now use

$$B(z)=\frac{4\mathrm{Li}_\nu(1-e^{-az})\mathrm{Li}_\nu(1-e^{-bz})(1-(-(1+\mu z))^{\frac{ab}{\mu}})(1+\mu z)^{\frac{ab(\xi+\eta)}{\mu}}}{((1+\mu z)^{\frac{a}{\mu}}+1)((1+\mu z)^{\frac{b}{\mu}}+1)^2}$$

to find that

$$B(z)=\left(\frac{2Li_\nu(1-e^{-az})}{(1+\mu z)^{\frac{a}{\mu}}+1}\right)(1+\mu z)^{\frac{ab\xi}{\mu}}\left(\frac{1-(-(1+\mu z))^{\frac{ab}{\mu}}}{(1+\mu z)^{\frac{b}{\mu}}+1}\right)$$
$$\times\left(\frac{2Li_\nu(1-e^{-bz})}{(1+\mu z)^{\frac{b}{\mu}}+1}\right)(1+\mu z)^{\frac{ab\eta}{\mu}}$$
$$=\sum_{j=0}^{\infty}G_j^{(\nu)}(b\xi;\mu)\frac{(az)^j}{j!}\sum_{r=0}^{\infty}\tau_r(a-1;\mu)\frac{(bz)^j}{j!}\sum_{i=0}^{\infty}G_i^{(\nu)}(a\eta;\mu)\frac{(bz)^i}{i!}$$
$$=\sum_{j=0}^{\infty}G_j^{(\nu)}(b\xi;\mu)\frac{(az)^j}{j!}\sum_{r=0}^{\infty}\sum_{i=0}^{r}\binom{r}{i}b^r\tau_i(a-1;\mu)G_{r-i,\mu}^{(\nu)}(a\eta)\frac{t^r}{r!}$$

$$B(z)=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}a^{j-r}b^rG_{j-r}^{(\nu)}(b\xi;\mu)\sum_{i=0}^{r}\binom{r}{i}\tau_i(a-1;\mu)G_{r-i,\mu}^{(\nu)}(a\eta)\right)\frac{z^j}{j!}. \quad (3.5)$$

Using a similar plan, we get

$$B(z)=\sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}a^rb^{j-r}G_{j-r}^{(\nu)}(a\xi;\mu)\sum_{i=0}^{r}\binom{r}{i}\tau_i(b-1;\mu)G_{r-i}^{(\nu)}(b\eta)\right)\frac{z^j}{j!}, \quad (3.6)$$

which yields the desired result.

4. Concluding remark

In this paper, we consider the modified degenerate poly-Genocchi polynomials are defined employing the following generating function

$$\frac{2Li_\nu(1-e^{-2z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi}{\mu}}=\sum_{j=0}^{\infty}G_{j,2}^{(\nu)}(\xi;\mu)\frac{z^j}{j!}. \quad (4.1)$$

Thus, by Eq. 4.1, we easily get $G_{0,2}^{(\nu)}(\xi;\mu)=0$. When $\xi=0$, $G_{j,2}^{(\nu)}(\mu)=G_{j,2}^{(\nu)}(0;\mu)$ are called the modified degenerate Genocchi numbers. For $\nu=1$, we note that

$$\frac{2Li_1(1-e^{-2z})}{(1+\mu z)^{\frac{1}{\mu}}+1}(1+\mu z)^{\frac{\xi}{\mu}}=\sum_{j=0}^{\infty}G_{j,2}(\xi;\mu)\frac{z^j}{j!}. \quad (4.2)$$

Hence
$G_{j,2}^{(1)}(\xi;\mu)=G_j(\xi;\mu),(j\geq 0)$.b

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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