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Integrability and wave solutions for fifth-order KdV type equation

A. A. Gaber [1, 2, *]

[1]*Department of Mathematics, College of Science and Humanities at Howtat Sudair, Majmaah University, Majmaah 11952, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt*

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ABSTRACT

Under investigation in this paper is the Kawahara equation, which is one of the fifth-order KdV types of equations. With the help of symbolic computation, we studied the integrability in the Painlevé property. Furthermore, Kruskal's transformation and Bäcklund transformation are used to obtain the exact wave solutions. Two wave solutions are obtained and figured to show the behavior of these solutions.

1. Introduction

As we known that there are many techniques to find the exact solutions for a given partial differential equation in the nonlinear science, such as the symmetry analysis, the bilinear form, Bäcklund transformation, and Darboux transformation, etc. (McLeod and Olver, 1983; Weiss et al., 1983; Ablowitz et al.,1980; Bhutani et al., 1995; Alagesan and Porsezian, 1996; Wang, 2010; Moatimid et al., 2012; Steeb and Euler, 1988; Hao et al., 2019; Wazwaz, 2012).

The Painlevé analysis has drawn attention in much of neoteric research, and in fact, this study has been well utilized by Weiss, Tabor and Carnevale (WTC) for the partial differential equations in 1983, plays a very important role to find many other integrable properties such as the Bäcklund transformations, Lax pair, Schwarzian form, and more new integrable models.

Consequently, we have, herein, utilized singular manifold expansion to obtain some special exact solutions of the Kawahara equation. Unlike the integrable cases of ODEs and PDEs, the procedure yields, for the non-integrable case, a consistency condition which, when exploited further, leads to certain special solutions. In this paper, we obtain the Kawahara equation:

$$u_t + \alpha u^2 u_x + \beta u_{xxx} + \gamma u_{xxxxx} = 0 \quad (1)$$

where u=u(x, t) denotes the unknown function, all the parameters α, β and γ ∈ R. This equation is one the type of fifth-order KdV equations, which described many physical phenomena, such as gravity-capillary waves on a shallow layer and magneto-sound propagation in plasma of Hunter and Scheurle (1988); the authors proved there are traveling wave solutions for a fifth-order partial differential equation, which describes water waves with surface tension. The paper Chen et al. (2009) is mainly concerned with the local well-posedness of the initial-value problems for the Kawahara and the modified Kawahara equations in Sobolev spaces.

2. Painlevé analysis

In this part, we study the Painlevé integrability of Eq. 1 following Weiss's algorithm (Weiss et al., 1983) of singularity analysis. To proceed with the Painlevé singularity analysis, set Eq. 1 to:

$$u(x,t) = \varphi^r(x,t) \sum_{j=0}^{\infty} u_j(x,t)\, \varphi^j(x,t) \quad (2)$$

where φ=φ(x, t) and u_j (j=0,1,2,...) are analytic functions in a neighborhood of the manifold determined by φ(t, x)=0. Further, r is an integer to be found. Inserting expansion Eq. 2 in Eq. 1 a leading order analysis uniquely determines the possible value of r analysis yields:

$$r = -2,\ u_0 = 6\sqrt{\frac{-10\gamma}{\alpha}}\varphi_x^2. \quad (3)$$

Substituting Eq. 2 into Eq. 1, we get:

* Corresponding Author.
Email Address: aagaber6@gmail.com
https://doi.org/10.21833/ijaas.2020.04.013
Corresponding author's ORCID profile:
https://orcid.org/0000-0001-6109-8898

$\sum_{j=0}^{\infty}(j-2)u_j\varphi^{j-3}\varphi_t+\sum_{j=0}^{\infty}u_{j,t}\,\varphi^{j-2}+$
$\beta(\sum_{j=0}^{\infty}(j-4)(j-3)(j-2)u_j\varphi^{j-5}\varphi_x+$
$3\sum_{j=0}^{\infty}(j-3)(j-2)u_{j,x}\varphi^{j-4}\varphi_x^2+$
$3\sum_{j=0}^{\infty}(j-2)u_{j,x}\varphi^{j-3}\varphi_{xx}+3\sum_{j=0}^{\infty}(j-2)$
$u_{j,xx}\varphi^{j-3}\varphi_x+5\sum_{j=0}^{\infty}(j-5)(j-4)$
$(j-3)(j-2)u_{j,x}\varphi^{j-6}+10\sum_{j=0}^{\infty}(j-5)$
$(j-4)(j-3)(j-2)u_j\varphi^{j-6}\varphi_x^3\varphi_{xx}+$
$30\sum_{j=0}^{\infty}(j-4)(j-3)(j-2)u_{j,x}\varphi^{j-5}\,\varphi_x^3\varphi_{xx}^2+$
$10\sum_{j=0}^{\infty}(j-4)(j-3)(j-2)u_j\varphi^{j-5}\varphi_x^2\varphi_{xxx}+$
$20\sum_{j=0}^{\infty}(j-3)(j-2)u_{j,x}\varphi^{j-4}\varphi_x\varphi_{xxx}+$
$10\sum_{j=0}^{\infty}(j-3)(j-2)u_j\varphi^{j-4}\varphi_{xx}\varphi_{xxx}+$
$10\sum_{j=0}^{\infty}(j-2)u_{j,xx}\varphi^{j-3}\varphi_{xxx}+$
$10\sum_{j=0}^{\infty}(j-3)(j-2)u_{j,xxx}\varphi^{j-4}\,\varphi_x^2+$
$10\sum_{j=0}^{\infty}(j-3)(j-2)u_{j,x}\varphi^{j-4}\,\varphi_x^2\varphi_{xxx}+$
$5\sum_{j=0}^{\infty}(j-3)(j-2)u_j\varphi^{j-4}\,\varphi_x\varphi_{xxxx}+$
$5\sum_{j=0}^{\infty}(j-2)u_{j,x}\varphi^{j-3}\,\varphi_{xxx}+\sum_{j=0}^{\infty}u_{j,xxxx}\varphi^{j-2}$
$5\sum_{j=0}^{\infty}(j-2)u_{j,xxxx}\varphi^{j-3}\,\varphi_x+$
$\sum_{j=0}^{\infty}(j-2)u_{j,xxxx}\varphi^{j-3}\,\varphi_{xxxxx})+$
$\alpha(\sum_{j=0}^{\infty}u_j\varphi^{j-2})(\sum_{j=0}^{\infty}u_{j,x}\varphi^{j-2}$
$+\sum_{j=0}^{\infty}(j-2)u_j\varphi^{j-3}\,\varphi_x)=0.\qquad(4)$

On collecting terms involving u_j (x,t) in Eq. 4, it is readily found that:

$$6u_j\sqrt{\frac{-10\gamma}{\alpha}}\varphi_x^2[(j+1)(j-6)(j-8)$$
$$(j^2-7j+30)]=F(u_{j-1},\ldots,u_{j,t}\ldots)\qquad(5)$$

for j = 0, 1, 2,… From Eq. 5 we find that j=-1, 6, and 8 are the resonances. Thus, Eq. 2 can be expressed in the following alternative form:

$$u(x,t)=u_0\varphi^{-2}+u_1\varphi^{-1}+u_2+u_3\varphi$$
$$+u_3\varphi^2+u_4\varphi^2+u_5\varphi^3+u_6\varphi^4\qquad(6)$$

where u_0, u_1 and u_2 are functions to be determined. The recursion relation for $u_j(t,x)$ is found to be,

$(j-6)u_{j-4}\varphi^{j-3}\varphi_t+u_{j-6,t}\varphi^{j-2}+$
$\beta((j-6)(j-5)(j-4)u_{j-2}\varphi^{j-5}\varphi_x+$
$3(j-6)(j-5)u_{j-3,x}\varphi^{j-4}\varphi_x^2+3(j-6)(j-5)$
$u_{j-3}\varphi^{j-4}\varphi_x\varphi_{xx}+3(j-6)u_{j-4,x}\varphi^{j-3}\varphi_{xx}$
$+3(j-6)u_{j-4,xx}\varphi^{j-3}\varphi_x+(j-6)u_{j-4}\varphi^{j-3}\varphi_x^3$
$+u_{j-5,xxx}\varphi^{j-2})+\gamma((j-6)(j-5)(j-4)$
$(j-3)(j-2)u_j\varphi^{j-7}\varphi_x^5+5(j-6)(j-5)(j-4)$
$(j-3)u_{j-1,x}\varphi^{j-6}\varphi_x^4+10(j-6)(j-5)(j-4)$
$(j-3)u_{j-1}\varphi^{j-6}\varphi_x^3\varphi_{xx}+30(j-6)(j-5)(j-4)$
$u_{j-2,x}\varphi^{j-5}\varphi_x^2\varphi_{xx}+15(j-6)(j-5)(j-4)$
$u_{j-2}\varphi^{j-5}\varphi_x^3\varphi_{xx}^2+15(j-6)(j-5)u_{j-3,x}\varphi^{j-4}\varphi_{xx}^2$
$+10(j-6)(j-5)(j-4)u_{j-2,xx}\varphi^{j-5}\varphi_x^3$
$30(j-6)(j-5)u_{j-3,xx}\varphi^{j-4}\varphi_x\varphi_{xx}+$
$10(j-6)(j-5)(j-4)u_{j-2}\varphi^{j-5}\varphi_x^2\varphi_{xxx}+$
$20(j-6)(j-5)u_{j-3,x}\varphi^{j-4}\varphi_x\varphi_{xxx}+10(j-6)$
$(j-5)u_{j-3}\varphi^{j-4}\varphi_{xx}\varphi_{xxx}+10(j-6)u_{j-4,xx}$
$\varphi^{j-3}\varphi_{xxx}+10(j-6)(j-5)u_{j-3,xxx}\varphi^{j-4}\varphi_x^2+$
$10(j-6)(j-5)u_{j-3,x}\varphi^{j-4}\varphi_x^2\varphi_{xxx}+5(j-6)$
$(j-5)u_{j-3}\varphi^{j-4}\varphi_x\varphi_{xxxx}+5(j-6)u_{j-4,x}\varphi^{j-3}$
$\varphi_{xxx}+5(j-6)u_{j-4,xxxx}\varphi^{j-3}\varphi_x+(j-2)$
$u_{j-4,xxxx}\varphi^{j-3}\varphi_{xxxxx}+u_{j-5,xxxx}\varphi^{j-2})+$
$\sum_{k=0}^{j}\sum_{m=0}^{k}u_{j-k}u_{k-m}\left(u_{m-1,x}+(m-2)\,u_m\varphi_x\right).\qquad(7)$

For j=0,1, 2, … the resonances are -1, 6, and 8, but -1, as previously referred to, is not admissible for these values od j. Further, $u_j(x,t)$ is an arbitrary function of x and t.

Assigning j=0, 1, 2 in Eq. 7, we get:

j=0

$$u_0=6\sqrt{\frac{-10\gamma}{\alpha}}\varphi_x^2\qquad(8)$$

j=1

$$u_1=\frac{600\gamma\varphi_x^4u_{0,x}+\alpha u_0^2u_{0,x}}{120\gamma\varphi_x^5+5\alpha u_0^2\varphi_x}\qquad(9)$$

j=2

$u_2=\dfrac{1}{4\alpha u_0^2\varphi_x}(-4\alpha u_1^2u_0\varphi_x+2\alpha u_0u_1u_{0,x}$
$+\alpha u_{0,x}u_{1,x}-24\beta u_0\varphi_x^3+\gamma(120u_{1,x}\varphi_x^3$
$+240u_1\varphi_x^4\varphi_{xx}-720u_1\varphi_x^2\varphi_{xx}-240u_{0,xx}\varphi_x^3$
$-240u_0\varphi_x^2\varphi_{xxx}\qquad(10)$

j=3

$u_3=\dfrac{1}{3\alpha u_0^2\varphi_x}(\alpha((-u_1^3-6u_0u_1u_2)\varphi_x+u_1^2u_{0,x}$
$+2u_0u_2u_{0,x}+2u_0u_1u_{1,x}+u_0^2u_{2,xx})$
$\sqrt{10c\gamma}(-6u_1\varphi_x^3+18u_{0,x}\varphi_x^2+18u_0\varphi_x\varphi_{xx})+$
$\gamma(-180u_{0,x}\varphi_x^2\varphi_{xx}-90u_1\varphi_x^2\varphi_{xx}+90u_{0,xx}\varphi_{xx}^2+$
$180u_{0,xx}\varphi_x\varphi_{xx}-60u_{1,xx}\varphi_x^3-60u_1\varphi_x^2\varphi_{xxx}+$
$120u_0\varphi_x\varphi_{xx}+60u_0\varphi_{xx}\varphi_{xxx}+60u_0^3\varphi_x^2+$
$30u_0\varphi_x\varphi_{xxxx}=0\qquad(11)$

j=4

$u_4=\dfrac{1}{2\alpha u_0^2\varphi_x}(-2u_0+\alpha((-2u_1^2u_2-$
$2u_0^2u_2-4u_0u_1u_3)\varphi_x+2u_2u_1u_0^2+2u_0u_3u_{0,x})$
$+\varphi_tu_1^2u_{1,x}+2u_0u_2u_{2,x}+u_0^2u_{3,x})+\beta(6u_{1,x}\varphi_x^2$
$6u_1\varphi_x\varphi_{xx}-6u_{0,x}\varphi_x-2u_0\varphi_{xxx})+\gamma(30u_{1,x}$
$60u_{1,xx}\varphi_x\varphi_{xx}+40u_{1,x}\varphi_x\varphi_{xxx}+20u_{0,xx}\varphi_{xx}$
$-20u_{0,xx}\varphi_{xx}\varphi_{xxx}-20u_{0,xx}\varphi_{xx}+10u_{1,xx}\varphi_x$
$-10u_{0,x}\varphi_{xxx}-10u_{0,xxx}\varphi_x-2u_0\varphi_{xxxxx}))\qquad(12)$

j=5

$u_5=\dfrac{1}{\alpha u_0^2\varphi_x}(-u_1\varphi_t+u_{0,x}+\alpha((-u_1u_2^2-u_3u_1^2$
$2u_0u_2u_3-2u_0u_1u_4)\varphi_x+u_0^2u_{0,x}+2u_1u_3u_{0,x}$
$+2u_1u_3u_{0,x}+2u_0u_4u_{0,x}+2u_0u_3+2u_1u_2u_{1,x}$
$+2u_0u_3u_{1,x}+u_1^2u_{2,x}+2u_0u_2u_{2,x}+2u_0u_1u_{3,x}$
$+u_0^2u_{4,x}+\beta(-3u_{1,x}\varphi_{xx}-3u_{1,xx}\varphi_x-u_1\varphi_{xxx}$
$+u_{0,xxx})+\gamma(-10u_{1,xx}\varphi_{xxx}-10u_{1,xxx}\varphi_{xx}-$
$5u_{1,x}\varphi_{xxx}-5u_{1,xxxx}\varphi_x-u_1\varphi_{xxxx}+u_{0,xxxx}))\qquad(13)$

j=6

$u_{1,t}+\alpha(2u_2u_3u_{0,x}+2u_1u_4u_{0,x}+2u_0u_5u_{0,x}$
$+u_1^2u_{2,x}+2u_1u_3u_{0,x}+2u_0u_4u_{0,x}+2u_1u_2u_{2,x}$
$+2u_0u_3u_{2,x}+u_1^2u_{3,x}+2u_0u_2u_{3,x}+2u_0u_1u_{4,x}$
$u_0^2u_{5,x})+\beta u_{1,xxx}+\gamma u_{1,xxxx}=0\qquad(14)$

Therefore, we know for resonance Laurent series (2) admits the sufficient number of arbitrary functions. So it is concluded that the Kawahara Eq. 1 possesses Painlevé property.

3. Exact solution of Kawahara equation

Case I: In this part, we would like to use Kruskal's transformation to obtain the soliton solution of Eq. 1, which can have written in the form:

$$\varphi(x,t)=x-ct\qquad(15)$$

Substituting Eq. 15 into Eqs. 9-13, proceeding as in earlier cases we get, for successive powers of φ^j,

$$u_0 = 6\sqrt{\frac{-10\gamma}{\alpha}},\ u_1 = \frac{\beta}{\gamma}\sqrt{\frac{-\gamma}{10\alpha}},$$
$$u_4 = \frac{\beta}{\gamma}\sqrt{\frac{\gamma}{10\alpha}}(\beta^2 + 10c\gamma),$$
$$u_3 = u_5 = u_6 = 0 \quad (16)$$

Further, the integrability condition (j=6) is satisfied identically. On combining Eqs. 16 and 6 we get:

$$u(x,t) = \sqrt{\frac{-\gamma}{10\alpha}}\left(\frac{60}{\left(x+\frac{\beta^2}{10\gamma}\right)^2} + \frac{\beta}{\gamma}\right) \quad (17)$$

where $c = \frac{-\beta^2}{10\gamma}$. Eq. 17 represents an exact solution to the Kawahara equation that, to our knowledge, is being reported for the first in literature.

Case II: After verifying the Painlevé property of the integrable dispersionless equation, we now proceed to obtain the other integrability properties like Bäcklund transformation. To construct the Bäcklund transformation, we now truncate the Laurent series at the constant level term, that is, $u_j = 0$, for j≥2, which gives:

$$u(x,t) = u_0\varphi^{-2} + u_1\varphi^{-1} + u_2 \quad (18)$$

In order to get the periodic solution of Eq. 1, we substitute a trial solution,

$$\varphi(t,x) = Exp(\theta(\eta)),\ \eta = x + c\,t, \quad (19)$$

into Eqs. 8, 9 and 18, where $u_2=0$, we get:

$$u(x,t) = \frac{6}{7}\sqrt{\frac{-10\gamma}{\alpha}}(5\theta^2 - 2\theta''). \quad (20)$$

To obtain the solution of Eq. 10, we put $5\theta^2 - 2\theta'' = k$ (where k is constant), then we utilized:

$$\theta = c_2 - \frac{2}{5}log[cosh(\sqrt{\frac{5\,k}{2}}(\eta + 2c_1)), \quad (21)$$

where c_1 and c_2 are constants. The periodic solution of Eq. 1 can be written in the following form:

$$u(x,t) = \frac{12}{7}\sqrt{\frac{-10\gamma}{\alpha}}k(tanh^2(\sqrt{\frac{5\,k}{2}}(x + c\,t$$
$$+2c_1) + sech^2(\sqrt{\frac{5\,k}{2}}(x + c\,t + 2c_1)) \quad (22)$$

4. Conclusions and discussions

In this paper, we have done the followings:

(1) We have investigated the integrability for the Kawahara equation via the Painlevé property, and two integrability conditions are obtained at j=6 and j=8.

(2) The integrability condition at J=6 was satisfied by Kruskal's transformation.

(3) We obtained two-wave solutions for the Kawahara equation. The first was inferred by Kruskal's transformation and the other by using the Bäcklund transformation.

Two wave solutions were shown in Fig. 1 and Fig. 3. Furthermore, the properties of the solution are shown in Fig. 2 and Fig. 4.

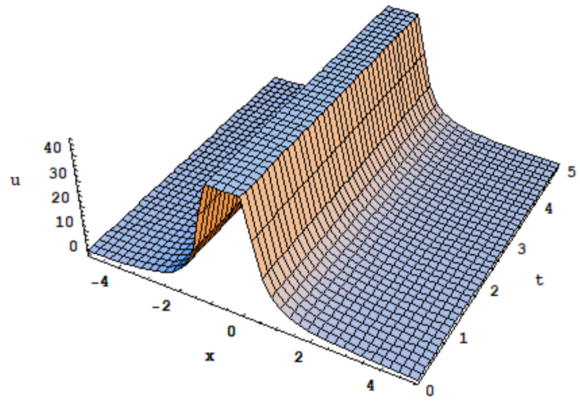

Fig. 1: Singular wave solution u(x,t) given by Eq. 17 with α=β=1, γ=-1

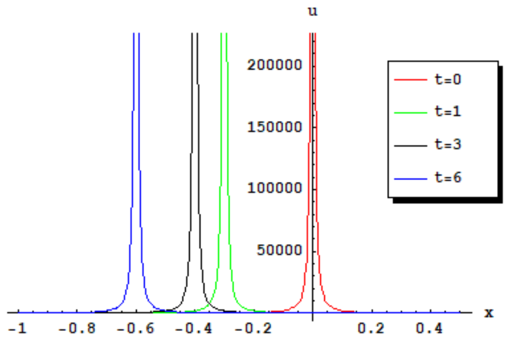

Fig. 2: Properties of the solution u(x,t) given by Eq. 17 for different values of t when α=β=1,γ=-1

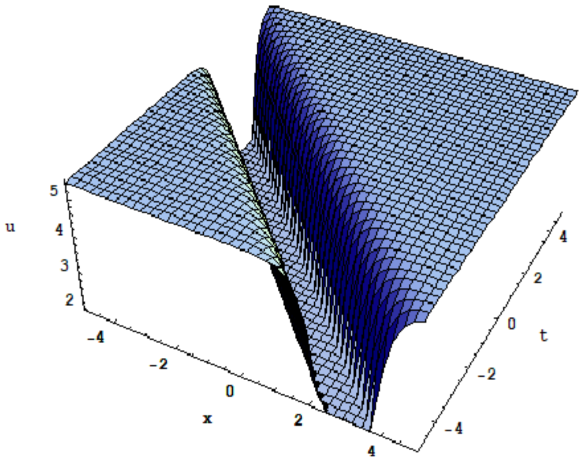

Fig. 3: Singular wave solution u(x,t) given by Eq. 22 with α=γ=c=c_1=k=1

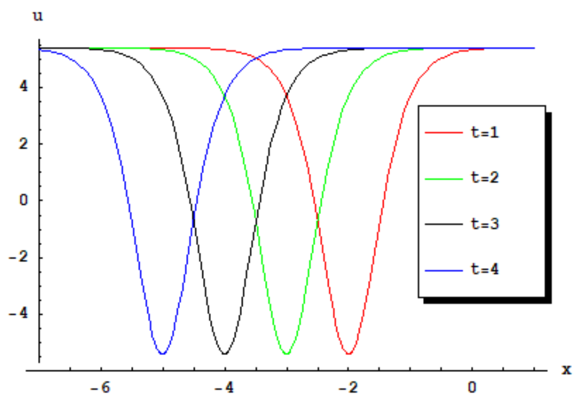

Fig. 4: Properties of the solution u(x,t) given by Eq. 22 for different values of t when α=γ=c=c1=k=1

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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