International Journal of Advanced and Applied Sciences, 7(3) 2020, Pages: 113-118

Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

A weighted version of Hermite-Hadamard type inequalities for strongly GA-convex functions

Nidhi Sharma [1], S. K. Mishra [1], A. Hamdi [2,*]

[1]*Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India*
[2]*Department of Mathematics, Statistics and Physics College of Arts and Sciences, Qatar University, Doha, Qatar*

ARTICLE INFO

Article history:
Received 11 October 2019
Received in revised form
6 January 2020
Accepted 7 January 2020

Keywords:
Convex function
Geometrically symmetric function
Strongly GA-convex function
Hermite Hadamard inequality
Holder inequality

ABSTRACT

In this paper, we have established new weighted Hermite-Hadamard type inequalities for strongly GA-convex functions. Those findings are obtained by using geometric symmetry of continuous positive mappings and differentiable mappings whose derivative in absolute value are strongly GA-convex. Some previous results are special cases of the results obtained in this paper.

1. Introduction

Karamardian (1969a) introduced strongly convex functions. However, we can find some references (Merentes and Nikodem, 2010; Nikodem and Páles, 2011) citing Polyak (1966) as being the pioneer to introduce this notion. Karamardian (1969b) investigated the class of scalar functions whose gradients are strongly monotone. It is well known that every continuously differentiable function is strongly monotone if its Jacobian matrix is strongly positive definite (Karamardian, 1969a).

Niculescu (2000) investigated the class of multiplicatively convex functions by replacing the arithmetic mean to the geometric mean. It is well known that every polynomial $p(x)$ with non-negative coefficients is a multiplicatively convex function on $[0,\infty)$. More generally, every real analytic function $\psi(x)=\sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients is a multiplicatively convex function on $(0,R)$, where R denotes the radius of convergence (Niculescu, 2000). Niculescu (2000) showed that a continuous function $\psi: G \subset (0,\infty) \rightarrow [0,\infty)$ is multiplicatively convex if and only if $x,y \in G \Rightarrow \psi(\sqrt{xy}) \leq \sqrt{\psi(x)\psi(y)}$.

Qi and Xi (2014) introduced a new concept of geometrically quasi-convex functions and established some integral inequalities of Hermite-Hadamard type for the function whose derivatives are of geometric quasi-convexity (Qi and Xi, 2014). Noor et al. (2017) introduced generalized geometrically convex functions and derived some basic inequalities related to generalize geometrically convex functions. Noor et al. (2017) also established new Hermite-Hadamard type inequalities for generalized geometrically convex functions. For more details, one can refer to (Latif, 2014; Niculescu and Persson, 2006; Noor et al., 2014a; 2014b; Shuang et al., 2013; Zhang et al., 2013).

Recently, Obeidat and Latif (2018) established some new weighted Hermite-Hadamard type inequalities for geometrically quasi-convex functions and also showed how we can use inequalities of Hermite-Hadamard type to obtain the inequalities for special means. For more details on Hermite-Hadamard inequalities, we refer the interested reader (Dragomir and Pearce, 2003; Latif, 2014; Shuang et al., 2013; Zhang et al., 2013; Qi et al., 2005).

Motivated by Noor et al. (2017) and Obeidat and Latif (2018), we establish some new weighted Hermite-Hadamard inequalities for strongly GA-convex functions by using geometric symmetry of a continuous positive mapping and a differentiable mapping whose derivatives in absolute value are strongly GA-convex.

2. Preliminaries

Let $\psi: [c,d] \rightarrow \mathbb{R}$ be a convex function with $c<d$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature.

* Corresponding Author.
Email Address: abhamdi@qu.edu.qa (A. Hamdi)
https://doi.org/10.21833/ijaas.2020.03.012
Corresponding author's ORCID profile:
https://orcid.org/0000-0003-1950-8907

$$\psi\left(\frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d \psi(x)dx \le \frac{\psi(c)+\psi(d)}{2}.$$

Definition 1: Let $G \subseteq \mathbb{R}_+ = (0,\infty)$. The set G is said to be geometrically convex set, if $x^{\delta}y^{1-\delta} \in G$, $\forall x, y \in G, \delta \in [0,1]$ (Niculescu, 2000).

Definition 2: A function $\psi: G \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be geometrically convex on G, if (Niculescu, 2000),

$$\psi(x^{\delta}y^{1-\delta}) \le \delta\psi(x) + (1-\delta)\psi(y), \quad \forall x, y \in G, \delta \in [0,1].$$

Definition 3: A function $\psi: G \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be geometrically symmetric with respect to $\sqrt{cd}$ if $\psi\left(\frac{cd}{x}\right) = \psi(x)$ for every $x \in G$ (Obeidat and Latif, 2018).

Definition 4: A function $\psi: G \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be strongly GA-convex with modulus $\mu > 0$, if (Turhan et al., 2018),

$$\psi(x^{\delta}y^{1-\delta}) \le \delta\psi(x) + (1-\delta)\psi(y) - \mu\delta(1-\delta)\parallel \ln y - \ln x \parallel^2, \quad \forall x, y \in G, \delta \in [0,1].$$

Lemma 1: For $0 < c < d$, we have (Obeidat and Latif, 2018):

$$(1)\ \triangle_1(c,d) = \int_0^1 \left|\ln\left(c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}}\right)\right|d\delta = \begin{cases} \frac{(\ln c)^2-3(\ln d)^2+2(\ln c)(\ln d)}{4(\ln d-\ln c)}, & \text{if } d \le 1, \\ \frac{-(\ln c)^2+3(\ln d)^2-2(\ln c)(\ln d)}{4(\ln d-\ln c)}, & \text{if } \sqrt{cd} \ge 1, \\ \frac{(\ln c)^2+5(\ln d)^2+2(\ln c)(\ln d)}{4(\ln d-\ln c)}, & \text{if } \sqrt{cd} < 1 < d. \end{cases}$$

$$(2)\ \triangle_2(c,d) = \int_0^1 c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}}\left|\ln\left(c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}}\right)\right|d\delta = \begin{cases} \frac{2[d-d\ln d-\sqrt{cd}+\sqrt{cd}\ln(\sqrt{cd})]}{\ln d-\ln c}, & \text{if } d \le 1, \\ \frac{2[-d+d\ln d+\sqrt{cd}-\sqrt{cd}\ln(\sqrt{cd})]}{\ln d-\ln c}, & \text{if } \sqrt{cd} \ge 1, \\ \frac{2[2-d+d\ln d-\sqrt{cd}+\sqrt{cd}\ln(\sqrt{cd})]}{\ln d-\ln c}, & \text{if } \sqrt{cd} < 1 < d. \end{cases}$$

Lemma 2: For $0 < c < d$, we have:

$$(1)\ \triangle_3(c,d) = \int_0^1 \delta c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}}\left|\ln\left(c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}}\right)\right|d\delta$$

$$= \begin{cases} \frac{4[d(-(\ln d)^2+(\ln d)(\ln\sqrt{cd})+2\ln d-\ln\sqrt{cd}-2)-\sqrt{cd}(\ln\sqrt{cd}-2)]}{(\ln d-\ln c)^2}, & \text{if } d \le 1, \\ \frac{4[d((\ln d)^2-(\ln d)(\ln\sqrt{cd})-2\ln d+\ln\sqrt{cd}+2)+\sqrt{cd}(\ln\sqrt{cd}-2)]}{(\ln d-\ln c)^2}, & \text{if } \sqrt{cd} \ge 1, \\ \frac{4[d((\ln d)^2-(\ln d)(\ln\sqrt{cd})-2\ln d+\ln\sqrt{cd}+2)+\sqrt{cd}(2-\ln\sqrt{cd})-2\ln\sqrt{cd}-4]}{(\ln d-\ln c)^2}, & \text{if } \sqrt{cd} < 1 < d. \end{cases}$$

$$(2)\ \triangle_4(c,d) = \int_0^1 \delta^2 c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}}\left|\ln\left(c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}}\right)\right|d\delta$$

$$= \begin{cases} \frac{8}{(\ln d-\ln c)^3}[d((\ln d)^2(-\ln d+2\ln\sqrt{cd}+3)-\ln d((\ln\sqrt{cd})^2 \\ +4\ln\sqrt{cd}+6)+(\ln\sqrt{cd})^2+4\ln\sqrt{cd}+6)-2\sqrt{cd}(3-\ln\sqrt{cd})], & \text{if } d \le 1, \\ \frac{8}{(\ln d-\ln c)^3}[d((\ln d)^2(\ln d-2\ln\sqrt{cd}-3)+\ln d((\ln\sqrt{cd})^2 \\ +4\ln\sqrt{cd}+6)-(\ln\sqrt{cd})^2-4\ln\sqrt{cd}-6)+2\sqrt{cd}(3-\ln\sqrt{cd})], & \text{if } \sqrt{cd} \ge 1, \\ \frac{8}{(\ln d-\ln c)^3}[d((\ln d)^3-(\ln\sqrt{cd})^2-3(\ln d)^2+6\ln d+\ln d(\ln\sqrt{cd})^2 \\ -2(\ln d)^2(\ln\sqrt{cd})+4\ln d\ln\sqrt{cd}-4\ln\sqrt{cd}-6) \\ +2(\ln\sqrt{cd})^2+2\sqrt{cd}\ln\sqrt{cd}+8\ln\sqrt{cd}-6\sqrt{cd}+12], & \text{if } \sqrt{cd} < 1 < d. \end{cases}$$

$$(3)\ \triangle_5(c,d) = \int_0^1 \delta\left|\ln\left(c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}}\right)\right|d\delta$$

$$= \begin{cases} \frac{2[-(\ln\sqrt{cd})^3-2(\ln d)^3+3(\ln\sqrt{cd})(\ln d)^2]}{3(\ln d-\ln c)^2}, & \text{if } d \le 1, \\ \frac{2[(\ln\sqrt{cd})^3+2(\ln d)^3-3(\ln\sqrt{cd})(\ln d)^2]}{3(\ln d-\ln c)^2}, & \text{if } \sqrt{cd} \ge 1, \\ \frac{2[-(\ln\sqrt{cd})^3+2(\ln d)^3-3(\ln\sqrt{cd})(\ln d)^2]}{3(\ln d-\ln c)^2}, & \text{if } \sqrt{cd} < 1 < d. \end{cases}$$

$$(4)\ \triangle_6(c,d) = \int_0^1 \delta^2\left|\ln\left(c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}}\right)\right|d\delta$$

$$= \begin{cases} \frac{2[(\ln\sqrt{cd})^4+8(\ln d)^3(\ln\sqrt{cd})-6(\ln d)^2(\ln\sqrt{cd})^2-3(\ln d)^4]}{3(\ln d-\ln c)^3}, & \text{if } d \le 1, \\ \frac{2[-(\ln\sqrt{cd})^4-8(\ln d)^3(\ln\sqrt{cd})+6(\ln d)^2(\ln\sqrt{cd})^2+3(\ln d)^4]}{3(\ln d-\ln c)^3}, & \text{if } \sqrt{cd} \ge 1, \\ \frac{2[(\ln\sqrt{cd})^4-8(\ln d)^3(\ln\sqrt{cd})+6(\ln d)^2(\ln\sqrt{cd})^2+3(\ln d)^4]}{3(\ln d-\ln c)^3}, & \text{if } \sqrt{cd} < 1 < d. \end{cases}$$

For simplicity, we will use the following notations throughout the manuscript:

$$\rho_1(\delta) = c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}} \ and\ \rho_2(\delta) = c^{\frac{1+\delta}{2}}d^{\frac{1-\delta}{2}}.$$

Lemma 3: Let $\psi: G \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on G^0 and $c, d \in G^0$ with $c < d$, and let $\lambda: [c,d] \to [0,\infty)$ be a continuous positive mapping and geometrically symmetric to $\sqrt{cd}$. If $\psi' \in L[c,d]$ and $\psi: G \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is geometrically symmetric with respect to $\sqrt{cd}$, the (Obeidat and Latif, 2018),

$$\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx$$
$$=\frac{(\ln d-\ln c)}{2(\ln d+\ln c)}\Big[\int_0^1\Big(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{(\ln x)\lambda(x)}{x}dx\Big)\rho_1(\delta)\ln(\rho_1(\delta))\psi'(\rho_1(\delta))d\delta$$
$$-\int_0^1\Big(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{(\ln x)\lambda(x)}{x}dx\Big)\rho_2(\delta)\ln(\rho_2(\delta))\psi'(\rho_2(\delta))d\delta\Big].$$

3. Main results

In this section, we will discuss our main results.

Theorem 1: Let $\psi: G\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable function on G^0 and $c,d\in G^0$ with $c<d$, and let $\lambda:[c,d]\to[0,\infty)$ be a continuous positive mapping and geometrically symmetric to $\sqrt{cd}$. If $\psi'\in L[c,d]$, $\psi: G\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ is geometrically symmetric with respect to $\sqrt{cd}$ and $|\psi'|$ is strongly GA-convex on $[c,d]$ with modulus $\mu>0$, then,

$$\Big|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\Big|$$
$$\leq\frac{(\ln d-\ln c)^2}{8}\parallel\lambda\parallel_\infty\Big[\frac{|\psi'(c)|}{2}(\Delta_2(c,d)+\Delta_2(d,c)-\Delta_3(c,d)+\Delta_3(d,c))$$
$$+\frac{|\psi'(d)|}{2}(\Delta_2(c,d)+\Delta_2(d,c)+\Delta_3(c,d)-\Delta_3(d,c))$$
$$-\frac{\mu}{4}\parallel\ln d-\ln c\parallel^2(\Delta_2(c,d)+\Delta_2(d,c)-\Delta_4(c,d)-\Delta_4(d,c))\Big],$$

where $\parallel\lambda\parallel_\infty=sup_{x\in[c,d]}|\lambda(x)|$.

Proof: For the proof of this theorem, we will use Lemma 3.

$$\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx$$
$$=\frac{(\ln d-\ln c)}{2(\ln d+\ln c)}\Big[\int_0^1\Big(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{(\ln x)\lambda(x)}{x}dx\Big)\rho_1(\delta)\ln(\rho_1(\delta))\psi'(\rho_1(\delta))d\delta$$
$$-\int_0^1\Big(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{(\ln x)\lambda(x)}{x}dx\Big)\rho_2(\delta)\ln(\rho_2(\delta))\psi'(\rho_2(\delta))d\delta\Big].$$

This implies,

$$\Big|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\Big|$$
$$\leq\frac{(\ln d-\ln c)}{2(\ln d+\ln c)}\parallel\lambda\parallel_\infty\Big[\int_0^1\Big(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{\ln x}{x}dx\Big)\rho_1(\delta)|\ln(\rho_1(\delta))||\psi'(\rho_1(\delta))|d\delta$$
$$+\int_0^1\Big(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{\ln x}{x}dx\Big)\rho_2(\delta)|\ln(\rho_2(\delta))||\psi'(\rho_2(\delta))|d\delta\Big].\quad(1)$$

Since $|\psi'|$ is strongly GA-convex function on $[c,d]$ with modulus $\mu>0$, we have:

$$|\psi'(\rho_1(\delta))|=|\psi'(c^{\frac{1-\delta}{2}}d^{\frac{1+\delta}{2}})|$$
$$\leq\frac{(1-\delta)}{2}|\psi'(c)|+\frac{(1+\delta)}{2}|\psi'(d)|-\frac{\mu}{4}(1-\delta)(1+\delta)\parallel\ln d-\ln c\parallel^2\quad(2)$$

and

$$|\psi'(\rho_2(\delta))|=|\psi'(c^{\frac{1+\delta}{2}}d^{\frac{(1-\delta)}{2}})|$$
$$\leq\frac{(1+\delta)}{2}|\psi'(c)|+\frac{(1-\delta)}{2}|\psi'(d)|-\frac{\mu}{4}(1+\delta)(1-\delta)\parallel\ln d-\ln c\parallel^2.\quad(3)$$

Using (2) and (3) in (1), we have:

$$\Big|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\Big|$$
$$\leq\frac{(\ln d-\ln c)^2}{8}\parallel\lambda\parallel_\infty\Big[\int_0^1\Big(\frac{(1-\delta)}{2}|\psi'(c)|+\frac{(1+\delta)}{2}|\psi'(d)|$$
$$-\frac{\mu}{4}(1-\delta^2)\parallel\ln d-\ln c\parallel^2\Big)\rho_1(\delta)|\ln(\rho_1(\delta))|d\delta$$
$$+\int_0^1\Big(\frac{(1+\delta)}{2}|\psi'(c)|+\frac{(1-\delta)}{2}|\psi'(d)|-\frac{\mu}{4}(1-\delta^2)\parallel\ln d-\ln c\parallel^2\Big)\rho_2(\delta)|\ln(\rho_2(\delta))|d\delta\Big].$$

By applying Lemma 1 and 2, we have:

$$\Big|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\Big|$$
$$\leq\frac{(\ln d-\ln c)^2}{8}\parallel\lambda\parallel_\infty\Big[\frac{|\psi'(c)|}{2}(\Delta_2(c,d)+\Delta_2(d,c)-\Delta_3(c,d)+\Delta_3(d,c))$$
$$+\frac{|\psi'(d)|}{2}(\Delta_2(c,d)+\Delta_2(d,c)+\Delta_3(c,d)-\Delta_3(d,c))$$
$$-\frac{\mu}{4}\parallel\ln d-\ln c\parallel^2(\Delta_2(c,d)+\Delta_2(d,c)-\Delta_4(c,d)-\Delta_4(d,c))\Big].$$

This completes the proof.

Corollary 1: If $\lambda(x)=\frac{1}{(\ln x)(\ln d-\ln c)}$, $\forall\ x\in[c,d]$ with $1<c<d<\infty$ in Theorem 1, then,

$$\Big|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}-\frac{1}{\ln d-\ln c}\int_c^d\frac{\psi(x)}{x}dx\Big|$$
$$\leq\frac{(\ln d-\ln c)}{8(\ln c)}\Big[\frac{|\psi'(c)|}{2}(\Delta_2(c,d)+\Delta_2(d,c)-\Delta_3(c,d)+\Delta_3(d,c))$$
$$+\frac{|\psi'(d)|}{2}(\Delta_2(c,d)+\Delta_2(d,c)+\Delta_3(c,d)-\Delta_3(d,c))$$
$$-\frac{\mu}{4}\parallel\ln d-\ln c\parallel^2(\Delta_2(c,d)+\Delta_2(d,c)-\Delta_4(c,d)-\Delta_4(d,c))\Big].$$

Corollary 2: If $\mu=0$ in Theorem 1, then,

$$\Big|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\Big|$$
$$\leq\frac{(\ln d-\ln c)^2}{8}\parallel\lambda\parallel_\infty\Big[\frac{|\psi'(c)|}{2}(\Delta_2(c,d)+\Delta_2(d,c)-\Delta_3(c,d)+\Delta_3(d,c))$$
$$+\frac{|\psi'(d)|}{2}(\Delta_2(c,d)+\Delta_2(d,c)+\Delta_3(c,d)-\Delta_3(d,c))\Big],$$

where $\parallel\lambda\parallel_\infty=sup_{x\in[c,d]}|\lambda(x)|$.

Remark 1: If $|\psi'|$ is geometrically quasi-convex, then the above theorem reduces to Theorem 1 of Obeidat and Latif (2018).

Theorem 2: Let $\psi: G\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable function on G^0 and $c,d\in G^0$ with $c<d$, and let $\lambda:[c,d]\to[0,\infty)$ be a continuous positive mapping and geometrically symmetric to $\sqrt{cd}$. If $\psi'\in L[c,d]$, $\psi: G\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ is geometrically symmetric with respect to $\sqrt{cd}$ and $|\psi'|^\alpha$ is strongly GA-convex on $[c,d]$ for $\alpha>1$ with modulus $\mu>0$, then,

$$\Big|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\Big|$$
$$\leq\frac{(\ln d-\ln c)^2}{8}\parallel\lambda\parallel_\infty\Big(\frac{\alpha-1}{\alpha}\Big)^{1-\frac{1}{\alpha}}\Big[\Big(\Delta_2(c^{\frac{\alpha}{\alpha-1}},d^{\frac{\alpha}{\alpha-1}})\Big)^{1-\frac{1}{\alpha}}$$

$\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_1(c,d)-\Delta_5(c,d))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_1(c,d)+\Delta_5(c,d))\right.$
$\left.-\frac{\mu}{4}\|\ln d-\ln c\|^2(\Delta_1(c,d)-\Delta_6(c,d))\right)^{1/\alpha}$
$+\left(\Delta_2(d^{\frac{\alpha}{\alpha-1}},c^{\frac{\alpha}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}$
$\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_1(d,c)+\Delta_5(d,c))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_1(d,c)-\Delta_5(d,c))\right.$
$\left.\left.-\frac{\mu}{4}\|\ln d-\ln c\|^2(\Delta_1(d,c)-\Delta_6(d,c))\right)^{1/\alpha}\right],$

where $\|\lambda\|_\infty=sup_{x\in[c,d]}|\lambda(x)|$.

Proof: From Lemma 3, we have,

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\right|$$
$$\leq\frac{(\ln d-\ln c)}{2(\ln d+\ln c)}\|\lambda\|_\infty\left[\int_0^1\left(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{\ln x}{x}dx\right)\rho_1(\delta)|\ln(\rho_1(\delta))||\psi'(\rho_1(\delta))|d\delta\right.$$
$$\left.+\int_0^1\left(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{\ln x}{x}dx\right)\rho_2(\delta)|\ln(\rho_2(\delta))||\psi'(\rho_2(\delta))|d\delta\right].$$

Applying Hölder's inequality, we have:

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\right|$$
$$\leq\frac{(\ln d-\ln c)^2}{8}\|\lambda\|_\infty\left[\left(\frac{\alpha-1}{\alpha}\int_0^1\rho_1^{\frac{\alpha}{\alpha-1}}(\delta)|\ln(\rho_1^{\frac{\alpha}{\alpha-1}}(\delta))|d\delta\right)^{1-\frac{1}{\alpha}}\right.$$
$$\times\left(\int_0^1|\ln(\rho_1(\delta))||\psi'(\rho_1(\delta))|^{\alpha}d\delta\right)^{1/\alpha}$$
$$+\left(\frac{\alpha-1}{\alpha}\int_0^1\rho_2^{\frac{\alpha}{\alpha-1}}(\delta)|\ln(\rho_2^{\frac{\alpha}{\alpha-1}}(\delta))|d\delta\right)^{1-\frac{1}{\alpha}}\times$$
$$\left.\left(\int_0^1|\ln(\rho_2(\delta))||\psi'(\rho_2(\delta))|^{\alpha}d\delta\right)^{1/\alpha}\right].$$

Using Lemma 1 and Lemma 2, and strong GA-convexity of $|\psi'|^{\alpha}$ on $[c,d]$ for $\alpha>1$ with modulus $\mu>0$, we have:

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\right|$$
$$\leq\frac{(\ln d-\ln c)^2}{8}\|\lambda\|_\infty\left[\left(\frac{\alpha-1}{\alpha}\Delta_2(c^{\frac{\alpha}{\alpha-1}},d^{\frac{\alpha}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}\right.$$
$$\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_1(c,d)-\Delta_5(c,d))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_1(c,d)+\Delta_5(c,d))\right.$$
$$\left.-\frac{\mu}{4}\|\ln d-\ln c\|^2(\Delta_1(c,d)-\Delta_6(c,d))\right)^{1/\alpha}$$
$$+\left(\frac{\alpha-1}{\alpha}\Delta_2(d^{\frac{\alpha}{\alpha-1}},c^{\frac{\alpha}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}$$
$$\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_1(d,c)+\Delta_5(d,c))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_1(d,c)-\Delta_5(d,c))\right.$$
$$\left.\left.-\frac{\mu}{4}\|\ln d-\ln c\|^2(\Delta_1(d,c)-\Delta_6(d,c))\right)^{1/\alpha}\right].$$

This completes the proof.

Corollary 3: If $\lambda(x)=\frac{1}{(\ln x)(\ln d-\ln c)}$, $\forall\ x\in[c,d]$ with $1<c<d<\infty$ in Theorem 2, then,

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}-\frac{1}{\ln d-\ln c}\int_c^d\frac{\psi(x)}{x}dx\right|$$
$$\leq\frac{(\ln d-\ln c)}{8(\ln c)}\left(\frac{\alpha-1}{\alpha}\right)^{1-\frac{1}{\alpha}}\left[(\Delta_2(c^{\frac{\alpha}{\alpha-1}},d^{\frac{\alpha}{\alpha-1}}))^{1-\frac{1}{\alpha}}\right.$$
$$\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_1(c,d)-\Delta_5(c,d))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_1(c,d)+\Delta_5(c,d))\right.$$
$$\left.-\frac{\mu}{4}\|\ln d-\ln c\|^2(\Delta_1(c,d)-\Delta_6(c,d))\right)^{1/\alpha}$$
$$+\left(\Delta_2(d^{\frac{\alpha}{\alpha-1}},c^{\frac{\alpha}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}$$
$$\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_1(d,c)+\Delta_5(d,c))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_1(d,c)-\Delta_5(d,c))\right.$$
$$\left.\left.-\frac{\mu}{4}\|\ln d-\ln c\|^2(\Delta_1(d,c)-\Delta_6(d,c))\right)^{1/\alpha}\right].$$

Corollary 4: If $\mu=0$ in Theorem 2, then,

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\right|$$
$$\leq\frac{(\ln d-\ln c)^2}{8}\|\lambda\|_\infty\left(\frac{\alpha-1}{\alpha}\right)^{1-\frac{1}{\alpha}}\left[\left(\Delta_2(c^{\frac{\alpha}{\alpha-1}},d^{\frac{\alpha}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}\right.$$
$$\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_1(c,d)-\Delta_5(c,d))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_1(c,d)+\Delta_5(c,d))\right)^{1/\alpha}$$
$$+\left(\Delta_2(d^{\frac{\alpha}{\alpha-1}},c^{\frac{\alpha}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}$$
$$\left.\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_1(d,c)+\Delta_5(d,c))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_1(d,c)-\Delta_5(d,c))\right)^{1/\alpha}\right],$$

where $\|\lambda\|_\infty=sup_{x\in[c,d]}|\lambda(x)|$.

Remark 2: If $|\psi'|^{\alpha}$ is geometrically quasi-convex, then the above theorem reduces to Theorem 2 of Obeidat and Latif (2018).

Theorem 3: Let $\psi:G\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable function on G^0 and $c,d\in G^0$ with $c<d$, and let $\lambda:[c,d]\to[0,\infty)$ be a continuous positive mapping and geometrically symmetric to $\sqrt{cd}$. If $\psi'\in L[c,d]$, $\psi:G\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ is geometrically symmetric with respect to $\sqrt{cd}$ and $|\psi'|^{\alpha}$ is strongly GA-convex on $[c,d]$ for $\alpha>1$ with modulus $\mu>0$ and $\alpha>l>0$, then,

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\right|$$
$$\leq\frac{(\ln d-\ln c)^2}{8}\|\lambda\|_\infty\left(\frac{\alpha-1}{\alpha-l}\right)^{1-\frac{1}{\alpha}}\left(\frac{1}{l}\right)^{1/\alpha}\left[\left(\Delta_2(c^{\frac{\alpha-l}{\alpha-1}},d^{\frac{\alpha-l}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}\right.$$
$$\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_2(c^l,d^l)-\Delta_3(c^l,d^l))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_2(c^l,d^l)+\Delta_3(c^l,d^l))-\frac{\mu}{4}\|\ln d-\ln c\|^2(\Delta_2(c^l,d^l)-\Delta_4(c^l,d^l))\right)^{1/\alpha}$$
$$+\left(\Delta_2(d^{\frac{\alpha-l}{\alpha-1}},c^{\frac{\alpha-l}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}$$
$$\times\left(\frac{|\psi'(c)|^{\alpha}}{2}(\Delta_2(d^l,c^l)+\Delta_3(d^l,c^l))+\frac{|\psi'(d)|^{\alpha}}{2}(\Delta_2(d^l,c^l)-\Delta_3(d^l,c^l))\right.$$
$$\left.\left.-\frac{\mu}{4}\|\ln d-\ln c\|^2(\Delta_2(d^l,c^l)-\Delta_4(d^l,c^l))\right)^{1/\alpha}\right],$$

where $\|\lambda\|_\infty=sup_{x\in[c,d]}|\lambda(x)|$.

Proof: From Lemma 3, we have,

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d\frac{(\ln x)\lambda(x)}{x}dx-\int_c^d\frac{(\ln x)\lambda(x)\psi(x)}{x}dx\right|$$
$$\leq\frac{(\ln d-\ln c)}{2(\ln d+\ln c)}\|\lambda\|_\infty\left[\int_0^1\left(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{\ln x}{x}dx\right)\rho_1(\delta)|\ln(\rho_1(\delta))||\psi'(\rho_1(\delta))|d\delta\right.$$
$$\left.+\int_0^1\left(\int_{\rho_2(\delta)}^{\rho_1(\delta)}\frac{\ln x}{x}dx\right)\rho_2(\delta)|\ln(\rho_2(\delta))||\psi'(\rho_2(\delta))|d\delta\right].\quad(4)$$

Applying Hölder's inequality in (??), we have:

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d \frac{(\ln x)\lambda(x)}{x}dx-\int_c^d \frac{(\ln x)\lambda(x)\psi(x)}{x}dx\right|$$
$$\leq \frac{(\ln d-\ln c)^2}{8}\parallel \lambda \parallel_\infty \left[\left(\frac{\alpha-1}{\alpha-l}\int_0^1 \rho_1^{\frac{\alpha-l}{\alpha-1}}(\delta)|\ln(\rho_1^{\frac{\alpha-l}{\alpha-1}}(\delta))|d\delta\right)^{1-\frac{1}{\alpha}}\right.$$
$$\times\left(\frac{1}{l}\int_0^1 \rho_1^l(\delta)|\ln(\rho_1^l(\delta))||\psi'(\rho_1(\delta))|^\alpha d\delta\right)^{1/\alpha}$$
$$+\left(\frac{\alpha-1}{\alpha-l}\int_0^1 \rho_2^{\frac{\alpha-l}{\alpha-1}}(\delta)|\ln(\rho_2^{\frac{\alpha-l}{\alpha-1}}(\delta))|d\delta\right)^{1-\frac{1}{\alpha}}$$
$$\left.\times\left(\frac{1}{l}\int_0^1 \rho_2^l(\delta)|\ln(\rho_2^l(\delta))||\psi'(\rho_2(\delta))|^\alpha d\delta\right)^{1/\alpha}\right].$$

Using Lemma 1 and Lemma 2, and strong GA-convexity of $|\psi'|^\alpha$ on $[c,d]$ for $\alpha>1$ with modulus $\mu>0$, we have:

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d \frac{(\ln x)\lambda(x)}{x}dx-\int_c^d \frac{(\ln x)\lambda(x)\psi(x)}{x}dx\right|$$
$$\leq \frac{(\ln d-\ln c)^2}{8}\parallel \lambda \parallel_\infty \left(\frac{\alpha-1}{\alpha-l}\right)^{1-\frac{1}{\alpha}}\left(\frac{1}{l}\right)^{1/\alpha}\left[(\Delta_2(c^{\frac{\alpha-l}{\alpha-1}},d^{\frac{\alpha-l}{\alpha-1}}))^{1-\frac{1}{\alpha}}\right.$$
$$\times\left(\frac{|\psi'(c)|^\alpha}{2}(\Delta_2(c^l,d^l)-\Delta_3(c^l,d^l))+\frac{|\psi'(d)|^\alpha}{2}(\Delta_2(c^l,d^l)+\Delta_3(c^l,d^l))\right.$$
$$\left.-\frac{\mu}{4}\parallel \ln d-\ln c\parallel^2(\Delta_2(c^l,d^l)-\Delta_4(c^l,d^l))\right)^{1/\alpha}$$
$$+\left(\Delta_2(d^{\frac{\alpha-l}{\alpha-1}},c^{\frac{\alpha-l}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}$$
$$\times\left(\frac{|\psi'(c)|^\alpha}{2}(\Delta_2(d^l,c^l)+\Delta_3(d^l,c^l))+\frac{|\psi'(d)|^\alpha}{2}(\Delta_2(d^l,c^l)-\Delta_3(d^l,c^l))\right.$$
$$\left.\left.-\frac{\mu}{4}\parallel \ln d-\ln c\parallel^2(\Delta_2(d^l,c^l)-\Delta_4(d^l,c^l))\right)^{1/\alpha}\right].$$

This completes the proof.

Corollary 5: If $\lambda(x)=\frac{1}{(\ln x)(\ln d-\ln c)}$, $\forall\ x\in[c,d]$ with $1<c<d<\infty$ in Theorem 3, then,

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}-\frac{1}{\ln d-\ln c}\int_c^d \frac{\psi(x)}{x}dx\right|$$
$$\leq \frac{(\ln d-\ln c)}{8(\ln c)}\parallel \lambda \parallel_\infty \left(\frac{\alpha-1}{\alpha-l}\right)^{1-\frac{1}{\alpha}}\left(\frac{1}{l}\right)^{1/\alpha}\left[(\Delta_2(c^{\frac{\alpha-l}{\alpha-1}},d^{\frac{\alpha-l}{\alpha-1}}))^{1-\frac{1}{\alpha}}\right.$$
$$\times\left(\frac{|\psi'(c)|^\alpha}{2}(\Delta_2(c^l,d^l)-\Delta_3(c^l,d^l))+\frac{|\psi'(d)|^\alpha}{2}(\Delta_2(c^l,d^l)+\Delta_3(c^l,d^l))\right.$$
$$\left.-\frac{\mu}{4}\parallel \ln d-\ln c\parallel^2(\Delta_2(c^l,d^l)-\Delta_4(c^l,d^l))\right)^{1/\alpha}$$
$$+\left(\Delta_2(d^{\frac{\alpha-l}{\alpha-1}},c^{\frac{\alpha-l}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}$$
$$\times\left(\frac{|\psi'(c)|^\alpha}{2}(\Delta_2(d^l,c^l)+\Delta_3(d^l,c^l))+\frac{|\psi'(d)|^\alpha}{2}(\Delta_2(d^l,c^l)-\Delta_3(d^l,c^l))\right.$$
$$\left.\left.-\frac{\mu}{4}\parallel \ln d-\ln c\parallel^2(\Delta_2(d^l,c^l)-\Delta_4(d^l,c^l))\right)^{1/\alpha}\right].$$

Corollary 6: If $\mu=0$ in Theorem 3, then,

$$\left|\frac{(\ln d)\psi(d)+(\ln c)\psi(c)}{\ln d+\ln c}\int_c^d \frac{(\ln x)\lambda(x)}{x}dx-\int_c^d \frac{(\ln x)\lambda(x)\psi(x)}{x}dx\right|$$
$$\leq \frac{(\ln d-\ln c)^2}{8}\parallel \lambda \parallel_\infty \left(\frac{\alpha-1}{\alpha-l}\right)^{1-\frac{1}{\alpha}}\left(\frac{1}{l}\right)^{1/\alpha}\left[\left(\Delta_2(c^{\frac{\alpha-l}{\alpha-1}},d^{\frac{\alpha-l}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}\right.$$
$$\times\left(\frac{|\psi'(c)|^\alpha}{2}(\Delta_2(c^l,d^l)-\Delta_3(c^l,d^l))+\frac{|\psi'(d)|^\alpha}{2}(\Delta_2(c^l,d^l)+\Delta_3(c^l,d^l))\right)^{1/\alpha}$$
$$+\left(\Delta_2(d^{\frac{\alpha-l}{\alpha-1}},c^{\frac{\alpha-l}{\alpha-1}})\right)^{1-\frac{1}{\alpha}}$$
$$\left.\times\left(\frac{|\psi'(c)|^\alpha}{2}(\Delta_2(d^l,c^l)+\Delta_3(d^l,c^l))+\frac{|\psi'(d)|^\alpha}{2}(\Delta_2(d^l,c^l)-\Delta_3(d^l,c^l))\right)^{1/\alpha}\right],$$

where $\parallel \lambda \parallel_\infty = sup_{x\in[c,d]}|\lambda(x)|$.

Remark 3: If $|\psi'|^\alpha$ is geometrically quasi convex, then the above theorem reduces to Theorem 3 of Obeidat and Latif (2018).

4. Conclusion

In this paper, some new weighted Hermite-Hadamard type inequalities for strongly GA-convex functions are obtained by using geometric symmetry of a continuous positive mapping and a differentiable mapping whose derivatives in absolute value are strongly GA-convex.

Funding

The research of the first author is supported by the UGC-BHU Research Fellowship, through sanction letter no: Ref No./Math/Res/Sept. 2017/117 and the second author is financially supported by the Department of Science and Technology, SERB, New Delhi, India, through grant no. MTR/2018/000121.

Acknowledgment

The authors would like to thank the anonymous reviewers for their helpful comments.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

References

Dragomir SS and Pearce C (2003). Selected topics on Hermite-Hadamard inequalities and applications. Mathematics Preprint Archive, 2003(3): 463-817.

Karamardian S (1969a). The nonlinear complementarity problem with applications, Part 1. Journal of Optimization Theory and Applications, 4(2): 87-98. https://doi.org/10.1007/BF00927414

Karamardian S (1969b). The nonlinear complementarity problem with applications, Part 2. Journal of Optimization Theory and Applications, 4(3): 167-181. https://doi.org/10.1007/BF00930577

Latif MA (2014). New Hermite–Hadamard type integral inequalities for GA-convex functions with applications. Analysis, 34(4): 379-389. https://doi.org/10.1515/anly-2012-1235

Merentes N and Nikodem K (2010). Remarks on strongly convex functions. Aequationes Mathematicae, 80(1-2): 193-199. https://doi.org/10.1007/s00010-010-0043-0

Niculescu C and Persson LE (2006). Convex functions and their applications. Springer, New York, USA. https://doi.org/10.1007/0-387-31077-0

Niculescu CP (2000). Convexity according to the geometric mean. Mathematical Inequalities and Applications, 3(2): 155-167. https://doi.org/10.7153/mia-03-19

Nikodem K and Páles Z (2011). Characterizations of inner product spaces by strongly convex functions. Banach Journal of Mathematical Analysis, 5(1): 83-87. https://doi.org/10.15352/bjma/1313362982

Noor MA, Noor KI, and Awan MU (2014a). Geometrically relative convex functions. Applied Mathematics and Information Sciences, 8(2): 607-616. https://doi.org/10.12785/amis/080218

Noor MA, Noor KI, and Awan MU (2014b). Some inequalities for geometrically arithmetically h-convex functions. Creative Mathematics and Informatics, 23(1): 91-98.

Noor MA, Noor KI, and Safdar F (2017). Generalized geometrically convex functions and inequalities. Journal of Inequalities and Applications, 2017: 202. https://doi.org/10.1186/s13660-017-1477-x **PMid:28932100 PMCid:PMC5575034**

Obeidat S and Latif MA (2018). Weighted version of Hermite–Hadamard type inequalities for geometrically quasi-convex functions and their applications. Journal of Inequalities and Applications, 2018: 307. https://doi.org/10.1186/s13660-018-1904-7 **PMid:30839800 PMCid:PMC6244744**

Polyak BT (1966). Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. Soviet Mathematics-Doklady, 7: 72-75.

Qi F and Xi BY (2014). Some Hermite–Hadamard type inequalities for geometrically quasi-convex functions. Proceedings-Mathematical Sciences, 124(3): 333-342. https://doi.org/10.1007/s12044-014-0182-7

Qi F, Wei ZL, and Yang Q (2005). Generalizations and refinements of Hermite-Hadamard's inequality. The Rocky Mountain Journal of Mathematics, 35(1): 235-251. https://doi.org/10.1216/rmjm/1181069779

Shuang Y, Yin HP, and Qi F (2013). Hermite–Hadamard type integral inequalities for geometric-arithmetically s-convex functions. Analysis International Mathematical Journal of Analysis and Its Applications, 33(2): 197-208. https://doi.org/10.1524/anly.2013.1192

Turhan S, Demirel AK, Maden S, and İşcan İ (2018). Hermite-Hadamard inequality for strongly GA-convex functions. Available online at: https://bit.ly/3bxhYnE

Zhang TY, Ji AP, and Qi F (2013). Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means. Le Matematiche, 68(1): 229-239.