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Separation axioms on fuzzy ideal topological spaces in Šostak's sense

Fahad Alsharari [1], Yaser M. Saber [1, 2, *]

[1]Department of Mathematics, College of Science and Human Studies, Hotat Sudair, Majmaah University, Majmaah 11952, Saudi Arabia

[2]Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt

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ABSTRACT

In the present paper, we introduce the notions of r-fuzzy ⋆-open and r-fuzzy ⋆-closed sets in Šostak's fuzzy topological spaces. Also, we study some properties of these notions. Moreover, we give the concept of fuzzy ideal ⋆-irresolute mapping in Šostak's fuzzy topological spaces. Finally, we study some kinds of separation axioms namely $\mathrm{r-FIR}_i$ where i={0.1.2.3} and $\mathrm{r-FIT}_j$ where j={1.2.$2\frac{1}{2}$.3.4} and the relations between them. Also, some of their characterizations and several of fundamental properties have been established.

1. Introduction

The concept of fuzzy topology was first defined by Chang (1968) and later redefined in a somewhat different way by Lowen (1976) and by Hutton and Reilly (1980). According to Hohle and Sostak (1999), in all these definitions, a fuzzy topology is a crisp subfamily of fuzzy sets and fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Therefore Šostak (1985) introduced a new definition of fuzzy topology. Later on, he developed the theory of fuzzy topological spaces in Šostak (1989). After that several authors (Chattopadhyay et al., 1992; Chattopadhyay and Samanta, 1993; El Gayyar et al., 1994; Kim and Ko, 2001; Ramadan, 1992; Ramadan et al., 2001) have introduced the smooth definition and studied smooth fuzzy topological spaces being unaware of Sostak (1985) works. In fuzzy topology, by introducing the notion of ideal, (Sarkar, 1997), and several other authors (Hatir and Jafari, 2007; Nasef and Mahmoud, 2002; Saber and Abdel-Sattar, 2014; Zahran et al., 2009; Alsharari and Saber, 2019; Saber and Alsharari, 2018) carried out such analysis. Throughout this paper, let X be a nonempty set, $I=[0.1]$, $I_0=(0.1]$ and I^X denote the set of all fuzzy subsets of X. For each $\alpha \in I$, $\underline{\alpha}(x)=\alpha$ for very $x \in X$.

A fuzzy point x_t is said to be quasi-coincident with a fuzzy set $\mathcal{A} \in I^X$ denoted by $x_t q \mathcal{A}$, if $t+\mathcal{A}(x)>1$. For $\mathcal{A}.\mathcal{B} \in I^X$, $\mathcal{A}$ is quasi-coincident with $\mathcal{B}$ denoted by $\mathcal{A} q \mathcal{B}$, if there exists $x \in X$ such that $\mathcal{A}(x)+\mathcal{B}(x)>1$. If $\mathcal{A}$ is not quasi-coincident with $\mathcal{B}$, we denoted $\mathcal{A} \bar{q} \mathcal{B}$ (Pao-Ming and Ying-Ming, 1980).

Definition 1.1: A fuzzy topological space (fts, for short) is an ordered pair $(X.\tau)$, where $\tau: I^X \rightarrow I$ is a mapping satisfying the following axioms (Šostak, 1985):

(01) $\tau(\underline{0})=\tau(\underline{1})=1$.
(02) $\tau(\mathcal{A} \cap \mathcal{B}) \geq \tau(\mathcal{A}) \cap \tau(\mathcal{B})$ for any $\mathcal{A}.\mathcal{B} \in I^X$.
(03) $\tau(\cup_{i\in\Gamma} \mathcal{A}_i) \geq \cap_{i\in\Gamma} \tau(\mathcal{A}_i)$ for any $\{\mathcal{A}_i\}_{i\in\Gamma} \in I^X$.

Theorem 1.1: Let $(X.\tau)$ be a fts. Then for each $r \in I_0$ and $\mathcal{A} \in I^X$. We define the operator $C_\tau: I^X \times I_0 \rightarrow I^X$ as follows (Chattopadhyay and Samanta, 1993):

$$C_\tau(\mathcal{A}.r)=\cap\{\mathcal{B} \in I^X | \mathcal{A} \leq \mathcal{B}.\tau(\underline{1}-\mathcal{B}) \geq r\}.$$

For each $\mathcal{A}.\mathcal{B} \in I^X$ and $r.s \in I_0$. the operator C_τ satisfies the following conditions:

(C1) $C_\tau(\underline{0}.r)=\underline{0}$.
(C2) $\mathcal{A} \leq C_\tau(\mathcal{A}.r)$.
(C3) $C_\tau(\mathcal{A}.r) \cup C_\tau(\mathcal{B}.r)=C_\tau(\mathcal{A} \cup \mathcal{B}.r)$.
(C4) $C_\tau(\mathcal{A}.r) \leq C_\tau(\mathcal{A}.s)$ if $r \leq s$.
(C5) $C_\tau(C_\tau(\mathcal{A}.r).r)=C_\tau(\mathcal{A}.r)$.
(C6) If $s=\cup\{r \in I_0 | C_\tau(\mathcal{A}.r)=\mathcal{A}\}$. then $C_\tau(\mathcal{A}.s)=\mathcal{A}$.

Theorem 1.2: Let $(X.\tau)$ be a fts. Then for each $r \in I_0$ and $\mathcal{A} \in I^X$. we define the operator $int_\tau: I^X \times I_0 \rightarrow I^X$ as follows (Hohle and Sostak, 1999):
$int_\tau(\mathcal{A}.r)=\cup\{\mathcal{B} \in I^X | \mathcal{A} \geq \mathcal{B}.\tau(\mathcal{B}) \geq r\}$.

* Corresponding Author.
Email Address: y.sber@mu.edu.sa (Y. M. Saber)
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Corresponding author's ORCID profile:
https://orcid.org/0000-0003-4795-3469

For each $\mathcal{A}.\mathcal{B} \in I^X$ and $r.s \in I_0$ the operator int_τ satisfies the following conditions:

(I1) $int_\tau(\underline{1} - \mathcal{A}.r) = \underline{1} - C_\tau(\mathcal{A}/r)$.
(I2) $int_\tau(\underline{1}.r) = \underline{1}$.
(I3) $int_\tau(\mathcal{A}.r) \leq \mathcal{A}$.
(I4) $int_\tau(\mathcal{A}.r) \cap int_\tau(\mathcal{B}.r) = int_\tau(\mathcal{A} \cap \mathcal{B}.r)$.
(I5) $int_\tau(\mathcal{A}.r) \geq int_\tau(\mathcal{A}.s)$ if $r \leq s$.
(I6) $int_\tau(int_\tau(\mathcal{A}.r).r) = int_\tau(\mathcal{A}.r)$.
(I7) If $s = \cup \{r \in I_0 | int_\tau(\mathcal{A}.r) = \mathcal{A}\}$.then $int_\tau(\mathcal{A}.s) = \mathcal{A}$.

Definition 1.2: A mapping $\mathcal{J}: I^X \to I$ is called a fuzzy ideal on X if it satisfies the following conditions (Saber and Abdel-Sattar, 2014):

(I_1) $\mathcal{J}(\underline{0}) = 1. \mathcal{J}(\underline{1}) = 0$.
(I_2) If $\mathcal{A} \leq \mathcal{B}$ then $\mathcal{J}(\mathcal{B}) \leq \mathcal{J}(\mathcal{A})$ for $\mathcal{A}.\mathcal{B} \in I^X$.
(I_3) $\mathcal{J}(\mathcal{A} \cup \mathcal{B}) \geq \mathcal{J}(\mathcal{A}) \cap \mathcal{J}(\mathcal{B})$ for $\mathcal{A}.\mathcal{B} \in I^X$.

If $\mathcal{J}_1$ and $\mathcal{J}_2$ are fuzzy ideals on X. we say that $\mathcal{J}_1$ is finer than $\mathcal{J}_2$ ($\mathcal{J}_2$ is coarser than $\mathcal{J}_1$), denoted by $\mathcal{J}_2 \leq \mathcal{J}_1$. iff $\mathcal{J}_1(\mathcal{A}) \leq \mathcal{J}_2(\mathcal{A})$ for $\mathcal{A} \in I^X$. The triple $(X.\tau.\mathcal{J})$ is called fuzzy ideal topological space (fits, for short). For $\alpha \in I_0$. $(X.\tau_\alpha.\mathcal{J}_\alpha)$ is fits in the sense of Sarkar.

Definition 1.3: Let $(X.\tau.\mathcal{J})$ be a fits and $\mathcal{A} \in I^X$. Then the r-fuzzy open local function $\mathcal{A}_r^\star(\tau.\mathcal{J})$ of $\mathcal{A}$ is the union of all fuzzy points x_t such that if $\mathcal{B} \in Q(x_t.r)$ and $\mathcal{J}(\mathcal{C}) \geq r$ then there is at least one $y \in X$ for which $\mathcal{B}(y) + \mathcal{A}(y) - 1 > \mathcal{C}(y)$ (Saber and Abdel-Sattar, 2014).

In other words, we say that a fuzzy set $\mathcal{A}$ is r-fuzzy open locally in $\mathcal{J}$ at x_t if there exists $\mathcal{B} \in Q(x_t.r)$ such that for every $y \in X$. $\mathcal{B}(y) + \mathcal{A}(y) - 1 \leq \mathcal{C}(y)$. for some $\mathcal{J}(\mathcal{C}) \geq r$. $\mathcal{A}_r^\star(\tau.\mathcal{J})$ is the set of fuzzy points at which $\mathcal{A}$ does not have the property r-fuzzy open locally.

We will occasionally write $\mathcal{A}_r^\star$ or $\mathcal{A}_r^\star(\mathcal{J})$ for $\mathcal{A}_r^\star(\tau.\mathcal{J})$ and it will cause no ambiguity.

Theorem 1.3: Let $(X.\tau)$ be a fts and $\mathcal{J}_1.\mathcal{J}_2$ be two fuzzy ideals of X. Then for each $r \in I_0$ and $\mathcal{A}.\mathcal{B} \in I^X$:

(1) If $\mathcal{A} \leq \mathcal{B}$ then $\mathcal{A}_r^\star \leq \mathcal{B}_r^\star$ (Saber and Abdel-Sattar, 2014).
(2) If $\mathcal{J}_1 \leq \mathcal{J}_2$ then $\mathcal{A}_r^\star(\mathcal{J}_1.\tau) \geq \mathcal{A}_r^\star(\mathcal{J}_2.\tau)$.
(3) $\mathcal{A}_r^\star = C_\tau(\mathcal{A}_r^\star.r) \leq C_\tau(\mathcal{A}.r)$.
(4) $(\mathcal{A}_r^\star)_r^\star \leq \mathcal{A}_r^\star$.
(5) $(\mathcal{A}_r^\star \cup \mathcal{B}_r^\star) = (\mathcal{A} \cup \mathcal{B})_r^\star$.
(6) If $\mathcal{J}(\mathcal{B}) \geq r$ then $(\mathcal{A} \cup \mathcal{B})_r^\star = \mathcal{A}_r^\star \cup \mathcal{B}_r^\star = \mathcal{A}_r^\star$.
(7) If $\tau(\mathcal{B}) \geq r$ then $(\mathcal{B} \cap \mathcal{A}_r^\star) \leq (\mathcal{B} \wedge \mathcal{A})_r^\star$.
(8) $(\mathcal{A}_r^\star \cap \mathcal{B}_r^\star) \geq (\mathcal{A} \cap \mathcal{B})_r^\star$.

Theorem 1.4: Let $(X.\tau.\mathcal{J})$ be a fits and $\{\mathcal{A}_i: i \in J\} \subset I^X$(Saber and Abdel-Sattar, 2014). Then:

(1) $(\cup (\mathcal{A}_i)_r^\star: \ i \in J) \leq (\cup \mathcal{A}_i: \ i \in J)_r^\star$.
(2) $(\cap\cap \mathcal{A}_i: \ i \in J)_r^\star \leq (\cap (\mathcal{A}_i)_r^\star: \ i \in J)$.

Remark 1.1: For each $(X.\tau.\mathcal{J})$ and $\mathcal{A} \in I^X$. we can define (Saber and Abdel-Sattar, 2014):

$Cl^\star(\mathcal{A}.r) = \mathcal{A} \vee \mathcal{A}_r^\star$.

$int^\star(\mathcal{A}.r) = \mathcal{A} \wedge [\underline{1} - (\underline{1} - \mathcal{A})_r^\star]$.

Clearly, $Cl^\star$ is a fuzzy closure operator and $\tau^\star(\mathcal{J})$ is the fuzzy topology generated by Cl^*; i.e.,

$\tau^\star(\mathcal{J})(\mathcal{A}) = \cup \{r | \ Cl^\star(\underline{1} - \mathcal{A}.r) = \underline{1} - \mathcal{A}\}$.

Now if,

$\mathcal{J} = \mathcal{J}^0$

then

$Cl^\star(\mathcal{A}.r) = \mathcal{A} \cup \mathcal{A}_r^\star = \mathcal{A} \cup C_\tau(\mathcal{A}.r) = C_\tau(\mathcal{A}.r)$.

For

$\mathcal{A} \in I^X$.

So,

$\tau^\star(\mathcal{J}^0) = \tau$.

Properties 1.1: Let $(X.\tau.\mathcal{J})$ be fits, $r \in I_0$ and $\mathcal{A} \in I^X$ (Saber and Abdel-Sattar, 2014). Then

(1) $int^\star(\mathcal{A} \cup \mathcal{B}.r) \leq int^\star(\mathcal{A}.r) \cup int^\star(\mathcal{B}.r)$.
(2) $int_\tau(\mathcal{A}.r) \leq int^\star(\mathcal{A}.r) \leq \mathcal{A} \leq Cl^\star(\mathcal{A}.r) \leq C_\tau(\mathcal{A}.r)$.
(3) $Cl^\star(\underline{1} - \mathcal{A}.r) = \underline{1} - int^\star(\mathcal{A}.r)$

and

$\underline{1} - Cl^\star(\mathcal{A}.r) = int^\star(\underline{1} - \mathcal{A}.r)$.
(4) $int^\star(\mathcal{A} \cap \mathcal{B}.r) = int^\star(\mathcal{A}.r) \cap int^\star(\mathcal{B}.r)$.

Definition 1.4: Let $(X.\tau)$ be a fts, $\mathcal{A} \in I^X$ and $r \in I_0$ (Ramadan et al., 2001). Then:

(1) $\mathcal{A}$ is called r-fuzzy semiopen set (r-FSO, for short) iff $\mathcal{A} \leq C_\tau(int_\tau(\mathcal{A}.r).r)$.
(2) $\mathcal{A}$ is called r-fuzzy semiclosed set (r-FSC, for short) iff $\underline{1} - \mathcal{A}$ is r-FSO set on X.
(3) $\mathcal{A}$ is called r-fuzzy β-open (r-FβO, for short) iff $\mathcal{A} \leq C_\tau(int_\tau(C_\tau(\mathcal{A}.r).r).r)$.

2. Fuzzy ideal ⋆-irresolute mapping

Definition 2.1: Let $(X.\tau.\mathcal{J})$ be a fits, $A \in I^X$ and $r \in I_0$. Then, $\mathcal{A}$ is called r-fuzzy ⋆-closed set iff $Cl^\star(\mathcal{A}.r) = \mathcal{A}$. The complement of a r-fuzzy ⋆-closed set is said to be a r-fuzzy ⋆-open set.

Proposition 2.1: Let $(X.\tau.\mathcal{J})$ be a fits and $r \in I_0$, $\mathcal{A} \in I^X$. Then:

(1) $\mathcal{A}$ is a r-fuzzy ⋆-closed set iff $\mathcal{A}_r^\star \leq \mathcal{A}$.
(2) $\mathcal{A}$ is a r-fuzzy ⋆-open set iff $\underline{1} - \mathcal{A}_r^\star \geq \underline{1} - \mathcal{A}$.
(3) If $\tau(\underline{1} - \mathcal{A}) \geq r$ (resp. $\tau(\mathcal{A}) \geq r, \mathcal{A}$),

then $\mathcal{A}$ is a r-fuzzy ⋆-closed (resp. r-fuzzy ⋆-open) set.
(4) If $\mathcal{A}$ is a r-FSC set (resp. r-FβC), then $int_\tau(\mathcal{A}^\star.r) \leq \mathcal{A}$ (resp. $int_\tau([int_\tau(\mathcal{A}.r)]^\star.r) \leq \mathcal{A}$).

Proof: The proofs of (1) and (2) are obvious from Definition 2.1. Let $\tau(\underline{1}-\mathcal{A})\geq r$. Then,

$$\mathcal{A}=C_\tau(\mathcal{A}.r)\geq Cl^\star(\mathcal{A}.r)=\mathcal{A}\vee\mathcal{A}_r^\star\geq\mathcal{A}_r^\star.$$

Hence $\mathcal{A}$ is a r-fuzzy ⋆-closed set. Other cases are similarly proved. Let $\mathcal{A}$ be a r-FSC set. Then,

$$\mathcal{A}\geq int_\tau(C_\tau(\mathcal{A}.r).r)\geq int_\tau(Cl^\star(\mathcal{A}.r).r)\ =int_\tau([\mathcal{A}\cup\mathcal{A}_r^\star].r)\geq int_\tau(\mathcal{A}_r^\star.r);$$

The other case is similarly proved.

Example 2.1: Define $\tau.\mathcal{I}:I^X\to I$ as follows:

$$\tau(\mathcal{B})=\begin{cases}1. & if\ \mathcal{B}\in\{\underline{0}.\underline{1}\}.\\ \frac{1}{2}. & if\ \mathcal{B}\in\{\underline{0\cdot 3}.\underline{0\cdot 7}\}.\\ 0. & otherwise.\end{cases}$$
$$\mathcal{I}(\mathcal{B})=\begin{cases}1. & if\ \mathcal{B}=\underline{0}.\\ \frac{1}{2}. & if\ \underline{0}\leq\mathcal{B}<\underline{0\cdot 3}.\\ 0. & otherwise\end{cases}$$

(1) $\underline{0\cdot 6}$ is a $\frac{1}{2}$-fuzzy ⋆-closed set but $\tau(\underline{1}-\underline{0\cdot 6})\neq\frac{1}{2}$.
(2) $\underline{0\cdot 2}\geq int_\tau(\underline{0\cdot 2}^\star_{\frac{1}{2}}.\frac{1}{2})=\underline{0}$, But $\underline{0\cdot 2}$ is not $\frac{1}{2}$-FSC set.

Lemma 2.1: Let $(X.\tau.\mathcal{I})$ be a fits. Then, the following properties hold:

(1) Any intersection of r-fuzzy ⋆-closed sets is a r-fuzzy ⋆-closed set.
(2) Any union of r-fuzzy ⋆-open sets is a r-fuzzy ⋆-open set.

Proof: (1) Let $\{\mathcal{A}_i\}_{i\in\Gamma}$ be a class of r-fuzzy ⋆-closed sets. Then for any $i\in\Gamma$. we have, $\mathcal{A}_i=Cl^\star(\mathcal{A}_i.r)$ and by Theorem 1.5(2), we have

$$\bigwedge_{i\in\Gamma}\mathcal{A}_i=\bigcap_{i\in\Gamma}Cl^\star(\mathcal{A}_i.r)=\bigcap_{i\in\Gamma}[\mathcal{A}_i\cup(\mathcal{A}_i)_r^\star]\geq\bigwedge_{i\in\Gamma}\mathcal{A}_i\cup\bigwedge_{i\in\Gamma}(\mathcal{A}_i)_r^\star\geq\bigcap_{i\in\Gamma}\mathcal{A}_i\cup(\bigcap_{i\in\Gamma}\mathcal{A}_i)_r^\star=Cl^\star(\bigcap_{i\in\Gamma}\mathcal{A}_i.r).$$

Hence, $\wedge_{i\in\Gamma}\mathcal{A}_i$ is a r-fuzzy ⋆-closed set. (2) It is easily proved in the same manner.

Lemma 2.2: Let $(X.\tau.\mathcal{I})$ be a fits, for each $r\in I_0$. Then

(1) For each r-fuzzy ⋆-open set $\mathcal{A}\in I^X$. $\mathcal{A}q\mathcal{B}$ iff $\mathcal{A}qCl^\star(\mathcal{B}.r)$.
(2) $x_tqCl^\star(\mathcal{B}.r)$ iff $\mathcal{A}q\mathcal{B}$. for each r-fuzzy ⋆-open set $\mathcal{A}\in I^X$ with $x_t\in\mathcal{A}$.

Proof: (1) Let $\mathcal{A}$ be a r-fuzzy ⋆-open set and $\mathcal{A}\overline{q}\mathcal{B}$. Then $\mathcal{B}\leq\underline{1}-\mathcal{A}$. Since $\mathcal{A}$ is a r-fuzzy ⋆-open set, $Cl^\star(\mathcal{B}.r)\leq Cl^\star(\underline{1}-\mathcal{A}.r)=\underline{1}-\mathcal{A}$. It follows that $\mathcal{A}\overline{q}Cl^\star(\mathcal{B}.r)$. (2) Let $x_tqCl^\star(\mathcal{B}.r)$. Then $\mathcal{A}qCl^\star(\mathcal{B}.r)$ with $x_t\in\mathcal{A}$. By (1), $\mathcal{B}q\mathcal{A}$, for each r-fuzzy ⋆-open set $\mathcal{A}\in I^X$. On the other hand, let $\mathcal{A}\overline{q}\mathcal{B}$. Then, $\mathcal{B}\leq\underline{1}-\mathcal{A}$. Since $\mathcal{A}$ is r-fuzzy ⋆-open set, $Cl^\star(\mathcal{B}.r)\leq Cl^\star(\underline{1}-\mathcal{A}.r)=\underline{1}-\mathcal{A}$ and $\mathcal{A}\overline{q}Cl^\star(\mathcal{B}.r)$. Since $x_t\in\mathcal{A}$. $x_t\overline{q}Cl^\star(\mathcal{B}.r)$.

Definition 2.2: Let $(X.\tau.\mathcal{I}_1)\to(Y.\eta.\mathcal{I}_2)$ be a mapping. Then, f is said to be:

(1) Fuzzy ideal ⋆-irresolute mapping (for short, $F\mathcal{I}$ ⋆-irresolute mapping) iff $f^{-1}(\mathcal{A})$ is r-fuzzy ⋆-open set in X. for each r-fuzzy ⋆-open set $\mathcal{A}$ in Y.
(2) Fuzzy ideal ⋆-irresolute open mapping (for short, $F\mathcal{I}$ ⋆-irresolute open mapping) iff $f(\mathcal{A})$ is r-fuzzy ⋆-open set in Y. for each r-fuzzy ⋆-open set $\mathcal{A}$ in X.
(3) Fuzzy ideal ⋆-irresolute closed mapping (for short, $F\mathcal{I}$ ⋆-irresolute closed mapping) iff $f(\mathcal{A})$ is r-fuzzy ⋆-closed set in Y. for each r-fuzzy ⋆-closed set $\mathcal{A}$ in X.

Theorem 2.1: Let $(X.\tau.\mathcal{I}_1)\to(Y.\eta.\mathcal{I}_2)$ be a mapping. Then the following statements are equivalent:

(1) f is $F\mathcal{I}$ ⋆-irresolute
(2) $f^{-1}(\mathcal{A})$ is a r-fuzzy ⋆-closed set, for each r-fuzzy ⋆-closed set $\mathcal{A}\in I^Y$.
(3) $f(Cl^\star(\mathcal{A}.r))\leq Cl^\star(f(\mathcal{A}).r)$ for each $\mathcal{A}\in I^X$, $r\in I_0$,
(4) $Cl^\star((f^{-1}(\mathcal{B}).r)\leq f^{-1}(Cl^\star(\mathcal{B}.r))$ for each $\mathcal{B}\in I^Y$, $r\in I_0$.

Proof: (1)⇒(2): Let $\mathcal{A}$ be a r-fuzzy ⋆-closed set in Y. Then, $\underline{1}-\mathcal{A}$ is a r-fuzzy ⋆-open set in Y. by (1), $f^{-1}(\underline{1}-\mathcal{A})$ is a r-fuzzy ⋆-open set. But, $f^{-1}(\underline{1}-\mathcal{A})=\underline{1}-f^{-1}(\mathcal{A})$. Then, $f^{-1}(\mathcal{A})$ is a r-fuzzy ⋆-closed set in X. (2)⇒(3): For each $\mathcal{A}\in I^X$ and $r\in I_0$. since $Cl^\star(Cl^\star(f(\mathcal{A}).r)=Cl^\star(f(\mathcal{A}).r)$. From Definition 2.1, $Cl^\star(f(\mathcal{A}).r)$ is a r-fuzzy ⋆-closed set in Y. By (2), $f^{-1}(Cl^\star(f(\mathcal{A}).r))$ is a r-fuzzy ⋆-closed set in X. Since,

$$\mathcal{A}\leq f^{-1}(f(\mathcal{A}))\leq f^{-1}(Cl^\star(f(\mathcal{A}).r)).$$

By Definition 2.1, we have,

$$Cl^\star(\mathcal{A}.r)\leq Cl^\star(f^{-1}(Cl^\star(f(\mathcal{A}).r)).r)=f^{-1}(Cl^\star(f(\mathcal{A}).r)).$$

Hence

$$f(Cl^\star(\mathcal{A}.r))\leq f(f^{-1}(Cl^\star(f(\mathcal{A}).r)))\leq Cl^\star(f(\mathcal{A}).r).$$

(3)⇒(4): For each $\mathcal{B}\in I^Y$ and $r\in I_0$. Put $\mathcal{A}=f^{-1}(\mathcal{B})$. By (3),

$$f(Cl^\star(f^{-1}(\mathcal{B}).r))\leq Cl^\star(f(f^{-1}(\mathcal{B})).r)\leq Cl^\star(\mathcal{B}.r).$$

It implies $Cl^\star(f^{-1}(\mathcal{B}).r)\leq f^{-1}(Cl^\star(\mathcal{B}.r))$. (4)⇒(1): Let $\mathcal{B}$ be a r-fuzzy ⋆-open set in Y. Then, $\underline{1}-\mathcal{B}$ is a r-fuzzy ⋆-closed set in Y. Hence, $Cl^\star(\underline{1}-\mathcal{B}.r)=\underline{1}-\mathcal{B}$, and by (4) we have,

$$f^{-1}(\underline{1}-\mathcal{B})=f^{-1}(Cl^\star(\underline{1}-\mathcal{B}.r))\geq Cl^\star(f^{-1}(\underline{1}-\mathcal{B}).r).$$

On the other hand, $f^{-1}(\underline{1}-\mathcal{B})\leq Cl^\star(f^{-1}(\underline{1}-\mathcal{B}).r)$. Thus, $f^{-1}(\underline{1}-\mathcal{B})=Cl^\star(f^{-1}(\underline{1}-\mathcal{B}).r)$. implies that

$f^{-1}(\underline{1}-\mathcal{B})$ is a r-fuzzy $\star$-closed set in X. Hence $f^{-1}(\mathcal{B})$ is a r-fuzzy $\star$-open set in X.

Theorem 2.2: Let $(X.\tau.\mathcal{I}_1) \to (Y.\eta.\mathcal{I}_2)$ be a mapping. Then the following statements are equivalent:

(1) f is $F\mathcal{I}$ $\star$-irresolute open.
(2) $f(int^\star(\mathcal{A}.r)) \leq int^\star(f(\mathcal{A}).r)$ for each $\mathcal{A} \in I^X$, $r \in I_0$
(3) $int^\star((f^{-1}(\mathcal{B}).r) \leq f^{-1}(int^\star(\mathcal{B}.r))$ for each $\mathcal{B} \in I^Y$, $r \in I_0$.
(4) For any $\mathcal{B} \in I^Y$ and any r-fuzzy $\star$-closed set $\mathcal{A} \in I^X$ with $f^{-1}(\mathcal{B}) \leq \mathcal{A}$. there exists r-fuzzy $\star$-closed $\mathcal{C} \in I^Y$ with $\mathcal{B} \leq \mathcal{C}$ such that $f^{-1}(\mathcal{C}) \leq \mathcal{A}$.

Proof: (1)⇒(2): For each $\mathcal{A} \in I^X$. $r \in I_0$ since $int^\star(\mathcal{A}.r) \leq \mathcal{A}$ from Remark 1.1, we have $f(int^\star(\mathcal{A}.r)) \leq f(\mathcal{A})$. By (1), $f(int^\star(\mathcal{A}.r))$ is a r-fuzzy $\star$-open set in Y. Hence,

$$f(int^\star(\mathcal{A}.r)) = int^\star(f(int^\star(\mathcal{A}.r))) \leq int^\star(f(\mathcal{A}).r)$$

(2)⇒(3): For each $\mathcal{B} \in I^Y$ and $r \in I_0$. Put $\mathcal{A} = f^{-1}(\mathcal{B})$ from (2),

$$f(int^\star(f^{-1}(\mathcal{B}),r)) \leq int^\star(f(f^{-1}(\mathcal{B})).r) \leq int^\star(\mathcal{B}.r).$$

It implies

$$int^\star(f^{-1}(\mathcal{B}).r) \leq f^{-1}\left(f(int^\star(f^{-1}(\mathcal{B}).r))\right) \leq f^{-1}(int^\star(\mathcal{B}.r)).$$

(3)⇒(4): Obvious.
(4)⇒(1): Let $\mathcal{D}$ be a r-fuzzy $\star$-open set in X. Put $\mathcal{B} = \underline{1} - f(\mathcal{D})$ and $\mathcal{A} = \underline{1} - \mathcal{D}$ such that $\mathcal{A}$ is a r-fuzzy $\star$-closed set in X. We obtain,

$$f^{-1}(\mathcal{B}) = f^{-1}(\underline{1} - f(\mathcal{D})) = \underline{1} - f^{-1}(f(\mathcal{D})) \leq \underline{1} - \mathcal{D} = \mathcal{A}.$$

From (4), there exists r-fuzzy $\star$-closed set $\mathcal{C} \in I^Y$ with $\mathcal{B} \leq \mathcal{C}$ such that $f^{-1}(\mathcal{C}) \leq \mathcal{A} = \underline{1} - \mathcal{D}$. It implies $\mathcal{D} \leq \underline{1} - f^{-1}(\mathcal{C}) = f^{-1}(\underline{1} - \mathcal{C})$. Thus,

$$f(\mathcal{D}) \leq f(f^{-1}(\underline{1} - \mathcal{C})) \leq \underline{1} - \mathcal{C}.$$

On the other hand, since, $\mathcal{B} \leq \mathcal{C}$. $f(\mathcal{D}) = \underline{1} - \mathcal{B} \geq \underline{1} - \mathcal{C}$. Hence, $f(\mathcal{D}) = \underline{1} - \mathcal{C}$. that is, $f(\mathcal{D})$ is a r-fuzzy $\star$-open set in Y. Theorem 2.3 is proved similarly to Theorem 2.2.

Theorem 2.3: Let $(X.\tau.\mathcal{I}_1) \to (Y.\eta.\mathcal{I}_2)$ be a mapping. Then the following statements are equivalent:

(1) f is $F\mathcal{I}$ $\star$-irresolute closed.
(2) $f(Cl^\star(\mathcal{B}.r)) \leq Cl^\star(f(\mathcal{B}).r)$ for each $\mathcal{B} \in I^X$, $r \in I_0$.

Theorem 2.4: Let $(X.\tau.\mathcal{I}_1) \to (Y.\eta.\mathcal{I}_2)$ be a bijective mapping. Then the following statements are equivalent:
(1) f is $F\mathcal{I}$ $\star$-irresolute closed.
(2) $Cl^\star(f^{-1}(\mathcal{A}).r) \leq f^{-1}(Cl^\star(\mathcal{A}.r))$ for each $\mathcal{A} \in I^Y.r \in I_0$.

Proof: (1) ⇒ (2): Let f be $F\mathcal{I}$ $\star$-irresolute closed. From Theorem 2.3 (2), for each $\mathcal{A} \in I^X$ and $r \in I_0$.

$$f(Cl^\star(\mathcal{B}.r)) \leq Cl^\star(f(\mathcal{B}).r).$$

For all $\mathcal{A} \in I^Y$. $r \in I_0$. put $\mathcal{B} = f^{-1}(\mathcal{A})$. since f is onto, $ff^{-1}(\mathcal{A}) = \mathcal{A}$. Thus,

$$f(Cl^\star(f^{-1}(\mathcal{A}).r)) \leq Cl^\star(f(f^{-1}(\mathcal{A})).r) = Cl^\star(\mathcal{A}).r).$$

Again since f is onto we have

$$Cl^\star(f^{-1}(\mathcal{A}).r) = f^{-1}(f(Cl^\star(f^{-1}(\mathcal{A}).r))) \leq f^{-1}(Cl^\star(\mathcal{A}.r)).$$

(2) ⇒ (1): Put $\mathcal{A} = f(\mathcal{B})$. Since f is injective,

$$Cl^\star(\mathcal{B}.r) = Cl^\star(f^{-1}(f(\mathcal{B})).r) \leq f^{-1}(Cl^\star(f(\mathcal{B}).r)).$$

Since f is onto,

$$f(Cl^\star(\mathcal{B}.r)) \leq f(f^{-1}(Cl^\star(f(\mathcal{B}).r))) = Cl^\star(f(\mathcal{B}).r).$$

3. Some types of separation axioms

Definition 3.1: Let $(X.\tau.\mathcal{I})$ be a fits and $r \in I_0$. Then X is said to be:

(1) $r-\mathrm{FIR}_0$ iff $x_t\overline{q}Cl^\star(y_s.r)$ implies $y_s\overline{q}Cl^\star(x_t.r)$ for any $x_t \neq y_s$.
(2) $r-\mathrm{FIR}_1$ iff $x_t\overline{q}Cl^\star(y_s.r)$ implies that there exist r-fuzzy $\star$-open sets $\mathcal{A}.\mathcal{B} \in I^X$ such that $x_t \in \mathcal{A}, y_s \in \mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$.
(3) $r-\mathrm{FIR}_2$ iff $x_t\overline{q}\mathcal{D} = Cl^\star(\mathcal{D}.r)$ implies there exist r-fuzzy $\star$-open sets $\mathcal{A},\mathcal{B} \in I^X$ such that $x_t \in \mathcal{A}, \mathcal{D} \leq \mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$.
(4)$r-\mathrm{FIR}_3$ iff $\mathcal{D}_1 = Cl^\star(\mathcal{D}_1.r)\overline{q}\mathcal{D}_2 = Cl^\star(\mathcal{D}_2.r)$ implies that there exist r-fuzzy $\star$-open sets $\mathcal{A}.\mathcal{B} \in I^X$ such that $\mathcal{D}_1 \leq \mathcal{A}.\mathcal{D}_2 \leq \mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$.
(5) $r-\mathrm{FIT}_1$ iff $x_t\overline{q}y_s$ implies that there exists a r-fuzzy $\star$-open set $\mathcal{A} \in I^X$ such that $x_t \in \mathcal{A}$ and $y_s\overline{q}\mathcal{A}$.
(6) $r-\mathrm{FIT}_2$ iff $x_t\overline{q}y_s$ implies that there exist r-fuzzy $\star$-open sets $\mathcal{A}.\mathcal{B} \in I^X$ such that $x_t \in \mathcal{A}$, $y_s \in \mathcal{B}$and $\mathcal{A}\overline{q}\mathcal{B}$.
(7) $r-\mathrm{FIT}_{2\frac{1}{2}}$ iff $x_t\overline{q}y_s$ implies that there exist r-fuzzy $\star$-open sets $\mathcal{A}.\mathcal{B} \in I^X$ such that $x_t \in \mathcal{A}$, $y_s \in \mathcal{B}$, and $Cl^\star(\mathcal{A}.r)\overline{q}Cl^\star(\mathcal{B}.r)$,
(8) $r-\mathrm{FIT}_3$ iff it is r-FIR $_2$ and $r-\mathrm{FIT}_1$,
(9) $r-\mathrm{FIT}_4$ iff it is r-FIR $_3$ and $r-\mathrm{FIT}_1$.

Theorem 3.1: Let $(X.\tau.\mathcal{I})$ be a fits and $r \in I_0$. Then the following statements are equivalent:

(1) $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIR}_0$,
(2) If $x_t\overline{q}\mathcal{A} = Cl^\star(\mathcal{A}.r)$, then there exists a r-fuzzy $\star$-open set $\mathcal{B} \in I^X$, such that $x_t\overline{q}\mathcal{B}$ and $\mathcal{A} \leq \mathcal{B}$.
(3) If $x_t\overline{q}\mathcal{A} = Cl^\star(\mathcal{A}.r)$, then $Cl^\star(x_t.r)\overline{q}\mathcal{A} = Cl^\star(\mathcal{A}.r)$,
(4) If $x_t\overline{q}Cl^\star(y_s.r)$, then $Cl^\star(x_t.r)\overline{q}Cl^\star(y_s.r)$.

Proof: (1)⇒(2): Let $x_t\overline{q}\mathcal{A} = Cl^\star(\mathcal{A}.r)$. Then $x_t\overline{q}Cl^\star(y_s.r)$ for each $y_s \in \mathcal{A}$. Since $(X.\tau.\mathcal{I})$ is

$r-\mathrm{FIR}_0$,$y_s\overline{q}Cl^\star(x_t.r)$, by Lemma 2.2(2), there exists a r-fuzzy ⋆-open set $\mathcal{D}\in I^X$, such that $x_t\overline{q}\mathcal{D}$ and $y_s\in\mathcal{D}$. Let $\mathcal{B}=\cup_{y_s\in\mathcal{A}}\{\mathcal{D}: x_t\overline{q}\mathcal{D}.\ y_s\in\mathcal{D}\}$, Form Lemma 2.1(1), $\mathcal{B}$ is a r-fuzzy ⋆-open set. Then $x_t\overline{q}\mathcal{B}$, $\mathcal{A}\le\mathcal{B}$.
(2)⇒(3): Let $x_t\overline{q}\mathcal{A}=Cl^\star(\mathcal{A}.r)$. Then by (2), there exists a r-fuzzy ⋆-open set $\mathcal{B}\in I^X$, such that $x_t\overline{q}\mathcal{B}$ and $\mathcal{A}\le\mathcal{B}$. Since $x_t\overline{q}\mathcal{B}$,

$$Cl^\star(x_t.r)\le Cl^\star(\underline{1}-\mathcal{B}.r)=\underline{1}-\mathcal{B}\le\underline{1}-\mathcal{A}.$$

Therefore,

$$Cl^\star(x_t.r)\overline{q}\mathcal{A}=Cl^\star(\mathcal{A}.r).$$

(3)⇒(4): Let $x_t\overline{q}Cl^\star(y_s.r)$. Then, $x_t\overline{q}Cl^\star(y_s.r)=Cl^\star(Cl^\star(y_s.r).r)$. By (3), $Cl^\star(x_t.r)\overline{q}Cl^\star(y_s.r)$.
(4)⇒(1): It is trivial.

Theorem 3.2: Let $(X.\tau.\mathcal{I})$ be a fits. Then:

(1) ($r-\mathrm{FIR}_3$ and $r-\mathrm{FIR}_0$) $\Rightarrow^{(a)}$ $r-\mathrm{FIR}_2$ $\Rightarrow^{(b)}$ $r-\mathrm{FIR}_1\Rightarrow^{(c)} r-\mathrm{FIR}_0$
(2) $r-\mathrm{FIT}_2\Rightarrow r-\mathrm{FIR}_1$.
(3) $r-\mathrm{FIT}_3\Rightarrow r-\mathrm{FIR}_2$.
(4) $r-\mathrm{FIT}_4\Rightarrow r-\mathrm{FIR}_3$.
(5) $r-\mathrm{FIT}_4\Rightarrow^{(a)} r-\mathrm{FIT}_3\Rightarrow^{(b)} r-\mathrm{FIT}_{2\frac{1}{2}}\Rightarrow^{(c)} r-\mathrm{FIT}_2$ $\Rightarrow^{(d)} r-\mathrm{FIT}_1$.

Proof: (1_a). Let $x_t\overline{q}\mathcal{D}=Cl^\star(\mathcal{D}.r)$. Then by Theorem 3.1 (3), $Cl^\star(x_t.r)\overline{q}\mathcal{D}=Cl^\star(\mathcal{D}.r)$. Since $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIR}_3$ and $Cl^\star(x_t.r)=Cl^\star(Cl^\star(x_t.r).r)$, there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B}\in I^X$such that $x_t\in Cl^\star(x_t.r)\le\mathcal{A}$. $\mathcal{D}\le\mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$. Hence $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIR}_2$. (1_b). For each $x_t\overline{q}Cl^\star(y_s.r)$. By $r-\mathrm{FIR}_2$ of X, there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B}\in I^X$ such that$x_t\in\mathcal{A}$, $y_s\in Cl^\star(y_s.r)\le\mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$. Hence $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIR}_1$. (1_c). Let $(X.\tau.\mathcal{I})$ be $r-\mathrm{FIR}_1$. Then for every $x_t\overline{q}Cl^\star(y_s.r)$ and $x_t\not\asymp y_s$ there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B}\in I^X$ such that $x_t\in\mathcal{A}$,$y_s\in\mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$. Therefore, $x_t\in\mathcal{A}\le\underline{1}-\mathcal{B}$. Since $\mathcal{B}$ is a r-fuzzy⋆-open set, $Cl^\star(x_t.r)\le Cl^\star(\underline{1}-\mathcal{B}.r)=\underline{1}-\mathcal{B}\le\underline{1}-y_s$. Hence $y_s\overline{q}Cl^\star(x_t.r)$ and $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIR}_0$.
(2) Let $x_t\overline{q}Cl^\star(y_s.r)$. Then $x_t\overline{q}y_s$. By $r-\mathrm{FIT}_2$ of X, there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B}\in I^X$ such that $x_t\in\mathcal{A}$, $y_s\in\mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$. Hence $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIR}_1$.
(3) and (4) are obvious.
(5_a). It is easily proved from (1).
(5_b). For each $x_t\overline{q}y_s$. Since r-FIR 2 and $r-\mathrm{FIT}_1$, are both in X, then there exists a r-fuzzy ⋆-open set $\mathcal{D}\in I^X$ such that $x_t\in\mathcal{D}$ and $y_s\overline{q}\mathcal{D}$. Then

$$x_t\in\mathcal{D}=int^\star(\mathcal{D}.r)\le int^\star(\underline{1}-y_s.r)=\underline{1}-Cl^\star(y_s.r).$$

Hence, $x_t\overline{q}Cl^\star(y_s.r)$, By $r-\mathrm{FIR}_2$ of X, there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B}\in I^X$ such that $x_t\in\mathcal{A}$ $Cl^\star(y_s.r)\le\mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$. Thus, $\mathcal{A}\le\underline{1}-\mathcal{B}$ and so,

$$Cl^\star(\mathcal{A}.r)\le Cl^\star(\underline{1}-\mathcal{B}.r)=\underline{1}-\mathcal{B}\le\underline{1}-Cl^\star(y_s.r).$$

It implies $Cl^\star(\mathcal{A}.r)\overline{q}Cl^\star(y_s.r)$ with $x_t\in\mathcal{A}$ and $y_s\in Cl^\star(y_s.r)$, Thus, $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIT}_{2\frac{1}{2}}$.

(5_c). Let $x_t\overline{q}y_s$. Then by $r-\mathrm{FIT}_{2\frac{1}{2}}$ of X, there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B}\in I^X$ such that $x_t\in\mathcal{A}$,$y_s\in\mathcal{B}$ and $Cl^\star(\mathcal{A}.r)\overline{q}Cl^\star(\mathcal{B}.r)$ implies that $\mathcal{A}\overline{q}\mathcal{B}$. Hence $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIT}_2$.
(5_d). Let $x_t\overline{q}y_s$. Then by $r-\mathrm{FIT}_2$ of X, there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B}\in I^X$ such that $x_t\in\mathcal{A}$,$y_s\in\mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$, Hence, $x_t\in\mathcal{A}$ and $y_s\overline{q}\mathcal{A}$. Thus, $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIT}_1$.

Example 3.1: Define $\tau_i.\mathcal{I}_i: I^X\to I$, $i=\{1.2\}$ as follows:

$$\tau_1(\mathcal{B})=\begin{cases}1. & if\ \mathcal{B}\in\{\underline{0}.\underline{1}\}.\\ \frac{1}{2}. & if\ \mathcal{B}=\underline{0\cdot 5}.\\ 0. & otherwise.\end{cases}$$
$$\mathcal{I}_1(\mathcal{B})=\begin{cases}1. & if\ \mathcal{B}=\underline{0}.\\ \frac{2}{3}. & if\ \underline{0}\le\mathcal{B}\le\underline{0\cdot 4}.\\ 0. & otherwise.\end{cases}$$
$$\tau_2(\mathcal{B})=\begin{cases}1. & if\ \mathcal{B}\in\{\underline{0}.\underline{1}\}.\\ \frac{1}{2}. & if\ \mathcal{B}=\underline{\alpha}.\ \underline{0}<\alpha<\underline{1}.\\ 0. & otherwise.\end{cases}$$
$$\mathcal{I}_2(\mathcal{B})=\begin{cases}1. & if\ \mathcal{B}=\underline{0}.\\ \frac{2}{3}. & if\ \underline{0}<B\le\underline{0\cdot 9}.\\ 0. & otherwise\end{cases}$$

(1) For $0<r\le 0\cdot 5$, $(X.\tau_1.\mathcal{I}_1)$ is $r-\mathrm{FIR}_0$ but it is not $r-\mathrm{FIR}_1$.
(2) For $0<r\le 0\cdot 5$, $(X.\tau_2.\mathcal{I}_2)$ is $r-\mathrm{FIT}_2$ but it is not $r-\mathrm{FIR}_{2\frac{1}{2}}$.

Theorem 3.3: An ifts $(X.\tau.\mathcal{I})$ is an $r-\mathrm{FIR}_1$ if and only if $x_t\overline{q}Cl^\star(y_s.r)$, there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B}\in I^X$ such that $\mathcal{A}\overline{q}\mathcal{B}$ and $Cl^\star(x_t.r)\le\mathcal{B}$. $Cl^\star(y_s.r)\le\mathcal{A}$.

Proof: (⇒) Let $x_t\overline{q}Cl^\star(y_s.r)$. Then by $r-\mathrm{FIR}_1$ of $(X.\tau.\mathcal{I})$, there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B}\in I^X$ such that $x_t\in\mathcal{A}$, $y_s\in\mathcal{B}$ and $\mathcal{A}\overline{q}\mathcal{B}$. Then, $x_t\overline{q}\underline{1}-\mathcal{A}$ implies that $Cl^\star(x_t.r)\le\underline{1}-\mathcal{A}\le\mathcal{B}$. Also, $y_s\overline{q}\underline{1}-\mathcal{B}$ implies that $Cl^\star(y_s.r)\le\underline{1}-\mathcal{B}\le\mathcal{A}$. (⇐) It is trivial.

Theorem 3.4: Let $(X.\tau.\mathcal{I})$ be a fits. Then the following statements are equivalent:

(1) $(X.\tau.\mathcal{I})$ is $r-\mathrm{FIR}_2$,
(2) If $x_t\in\mathcal{A}$ and $\mathcal{A}$ is a r-fuzzy ⋆-open set, then there exists a r-fuzzy ⋆-open set $\mathcal{B}\in I^X$, such that $x_t\in\mathcal{B}\le Cl^\star(\mathcal{B}.r)\le\mathcal{A}$.
(3) If $x_t\overline{q}\mathcal{A}=Cl^\star(\mathcal{A}.r)$, then there exists r-fuzzy ⋆-open sets $\mathcal{B}_i\in I^X$, $i=\{1.2\}$ such that $x_t\in\mathcal{B}_1$, $\mathcal{A}\le\mathcal{B}_2$ and $Cl^\star(\mathcal{B}_1.r)\overline{q}Cl^\star(\mathcal{B}_2.r)$.

Proof: (1)⇒(2): Let $\mathcal{A}$ be a r-fuzzy ⋆-open set and $x_t\in\mathcal{A}$. Then $x_t\overline{q}\underline{1}-\mathcal{A}$, By $r-\mathrm{FIR}_2$ of X. There exist r-fuzzy ⋆-open sets, $\mathcal{B}.\mathcal{C}\in I^X$ such that $x_t\in\mathcal{B}$, $\underline{1}-\mathcal{A}\le\mathcal{C}$ and $\mathcal{B}\overline{q}\mathcal{C}$. Hence, $x_t\in\mathcal{B}\le Cl^\star(\mathcal{B}.r)\le\underline{1}-\mathcal{C}\le\mathcal{A}$.

(2)⇒(3): Let $x_t \overline{q} \mathcal{A} = Cl^{\star}(\mathcal{A}.r)$. Then $x_t \in \underline{1} - \mathcal{A}$. By (2), there exists a r-fuzzy ⋆-open set $\mathcal{B} \in I^X$, such that $x_t \in \mathcal{B} \leq Cl^{\star}(\mathcal{B}.r) \leq \underline{1} - \mathcal{A}$. Since $\mathcal{B}$ is a r-fuzzy ⋆-open set and $x_t \in \mathcal{B}$. Again by (2), then there exists a r-fuzzy ⋆-open set $\mathcal{B}_1 \in I^X$, such that

$$x_t \in \mathcal{B}_1 \leq Cl^{\star}(\mathcal{B}_1.r) \leq \mathcal{B} \leq Cl^{\star}(\mathcal{B}.r) \leq \underline{1} - \mathcal{A}.$$

It implies that

$$\mathcal{A} \leq \underline{1} - Cl^{\star}(\mathcal{B}.r) = int^{\star}(\underline{1} - \mathcal{B}.r) \leq \underline{1} - \mathcal{B}.$$

Put

$$\mathcal{B}_2 = int^{\star}(\underline{1} - \mathcal{B}.r).$$

Then,

$$Cl^{\star}(\mathcal{B}_2.r) \leq \underline{1} - \mathcal{B} \leq \underline{1} - Cl^{\star}(\mathcal{B}_1.r),$$

that is,

$$Cl^{\star}(\mathcal{B}_1.r) \overline{q} Cl^{\star}(\mathcal{B}_2.r).$$

(3)⇒(1): Let $x_t \overline{q} \mathcal{A} = Cl^{\star}(\mathcal{A}.r)$. Then by (3), there exists r-fuzzy ⋆-open sets $\mathcal{B}_i \in I^X$, $i = \{1.2\}$ such that $x_t \in \mathcal{B}_1$, $\mathcal{A} \leq \mathcal{B}_2$ and $Cl^{\star}(\mathcal{B}_1.r) \overline{q} Cl^{\star}(\mathcal{B}_2.r)$. Hence, $\mathcal{B}_1 \overline{q} \mathcal{B}_2$ and $(X.\tau.\mathcal{I})$ is $\mathrm{r-FIR}_2$. The following theorem is similarly proved as in Theorem 3.4.

Theorem 3.5: Let $(X.\tau.\mathcal{I})$ be a fits. Then the following statements are equivalent:

(1) $(X.\tau.\mathcal{I})$ is $\mathrm{r-FIR}_3$,
(2) If $\mathcal{A}_1 \overline{q} \mathcal{A}_2$ and $\mathcal{A}_1.\mathcal{A}_2$ are r-fuzzy ⋆-closed sets, then there exists a r-fuzzy ⋆-open set $\mathcal{B} \in I^X$, such that $\mathcal{A}_1 \leq \mathcal{B}$ and $Cl^{\star}(\mathcal{B}.r) \overline{q} \mathcal{A}_2$.
(3) For any $\mathcal{A}_1 \leq \mathcal{A}_2$ and $\mathcal{A}_1$ is a r-fuzzy ⋆-open set, $\mathcal{A}_2$ is a r-fuzzy ⋆-closed set, there exists a r-fuzzy ⋆-open set $\mathcal{B} \in I^X$ such that $\mathcal{A}_1 \leq \mathcal{B} \leq Cl^{\star}(\mathcal{B}.r) \leq \mathcal{A}_2$.

Theorem 3.6: Let $f: (X.\tau.\mathcal{I}) \rightarrow (Y.\eta.\mathcal{I})$ be $F\mathcal{I}$ ⋆-irresolute, bijective, $F\mathcal{I}$ ⋆-irresolute open mapping and $(X.\tau.\mathcal{I})$ is $\mathrm{r-FIR}_2$ (resp. $\mathrm{r-FIR}_3$). Then $(Y.\tau.\mathcal{I})$ is $\mathrm{r-FIR}_2$ (resp. $\mathrm{r-FIR}_3$).

Proof: Let $y_t \overline{q} \mathcal{D} = Cl^{\star}(\mathcal{D}.r)$. Then, by definition 2.1, $\mathcal{D}$ is a r-fuzzy ⋆-closed set in Y. By Theorem 2.1(2), $f^{-1}(\mathcal{D})$ is a r-fuzzy ⋆-closed set in X. Put $y_s = f(x_s)$. Then $x_s \overline{q} f^{-1}(\mathcal{D})$. By r-FIR $_2$ of X, there exist r-fuzzy ⋆-open sets $\mathcal{A}.\mathcal{B} \in I^X$ such that $x_s \in \mathcal{A}$, $f^{-1}(\mathcal{D}) \leq \mathcal{B}$ and $\mathcal{A} \overline{q} \mathcal{B}$. Since f is bijective and $F\mathcal{I}$ ⋆-irresolute open, $y_s \in f(\mathcal{A})$, $\mathcal{D} \leq f(f^{-1}(\mathcal{D})) \leq f(\mathcal{B})$ and $f(\mathcal{A}) \overline{q} f(\mathcal{B})$. Hence $(Y.\tau.\mathcal{I})$ is $\mathrm{r-FIR}_2$. The other case is similarly proved.

4. Conclusion

The study introduced the nations r-fuzzy ⋆-open and r-fuzzy ⋆-closed sets in Šostak's fuzzy topological spaces along with the examination of some of their properties. It also inspected the concept of fuzzy ideal ⋆- irresolute mapping. Finally, it investigated some kinds of separation axioms namely $\mathrm{r} - FIR_i$ where i={0, 1, 2, 3} and $\mathrm{r} - FIR_j$ where j={1, 2, $2\frac{1}{2}$, 3, 4} as well as some of their characterizations and fundamental properties.

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Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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