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On paranorm I-convergent double sequence spaces defined by a compact operator

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ABSTRACT

In this paper, we extend the concept of ideal convergence of sequences in metric spaces. Recently, the concept of ideal convergent double sequence spaces defined by a compact operator. Motivated by this, we introduce some ideal convergent double sequence spaces with the help of compact operator T on the real space $\mathbb{R}$ and a bounded double sequence $p = (p_{ij})$ of positive real numbers. We examine some basic properties and prove some inclusions relations on these new defined sequence spaces.

1. Introduction

Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the sets of all natural, real, and complex numbers respectively. We denote ${}_2\omega$ showing the space of all real or complex double sequences.The ${}_2l_\infty$, ${}_2c$ and ${}_2c_0$ be denoted the Banach spaces of bounded, convergent and null double sequences of reals, respectively with the norm

$$\| x \| = \sup_{ij} |x_{ij}|.$$

As a generalization of usual convergence, the concept of statistical convergent was first introduced by Fast (1951) and also independently by Buck (1953) and Schoenberg (1959) for real and complex sequences. Later on, it was further investigated from a sequence space point of view and linked with the summability theory by Fridy (1985), Šalát (1980), and many other authors. After that, the notion of ideal convergence (I-convergence) was introduced and studied by Kostyrko et al. (2000, 2005). Later on, it was studied by Šalát et al. (2004, 2005), Tripathy and Hazarika (2009, 2011), Khan et al. (2014, 2015, 2017), Demirci (2001), Gürdal and Sahiner (2008), Nabiev et al. (2007), Sahiner et al. (2007, 2011), and the references therein.

Definition 1.1: Let X and Y be two normed linear spaces. An operator $T: X \to Y$ is said to be a compact linear operator (or completely continuous linear operator), if:

1. T is linear
2. T maps every bounded sequence (x_k) in X onto a sequence $T(x_k)$ in Y which has a convergent subsequence.

The set of all compact linear operator $\mathcal{C}(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$ and $\mathcal{C}(X, Y)$ is a Banach space if Y is a Banach space (Kreyszig, 1978). Following Başar and Altay (2003) and Şengönül (2007) were introduce the double sequence spaces ${}_2S$ and ${}_2S_0$ with the help of compact operator T on $\mathbb{R}$ as follows:

$${}_2S = \{x = (x_{ij}) \in {}_2l_\infty : T(x) \in {}_2c\}$$
$${}_2S_0 = \{x = (x_{ij}) \in {}_2l_\infty : T(x) \in {}_2c_0\}.$$

Here we give some preliminaries about the notion of I-convergence.

Definition 1.2: Let $\mathbb{N} \times \mathbb{N}$ be a non empty set. Then, a family of sets $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be an ideal in X if

1. $\emptyset \in I$;
2. I is additive; that is, $A, B \in I \Rightarrow A \cup B \in I$;
3. I is hereditary that is, $A \in I, B \subseteq A \Rightarrow B \in I$.

An Ideal $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is called non trivial if $I \neq 2^{\mathbb{N}\times\mathbb{N}}$. A non trivial ideal $I \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is called admissible if:

$$\{\{x\}: x \in \mathbb{N} \times \mathbb{N}\} \subseteq I.$$

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A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Definition 1.3: A non-empty family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is said to be filter on X if and only if

1. $\emptyset \notin \mathcal{F}$;
2. for, A, B $\in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$;
3. for each A$\in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.

For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I. That is,

$$\mathcal{F}(I) = \{K \subseteq \mathbb{N}\times\mathbb{N}: K^c \in I, where \quad K^c = \mathbb{N}\times\mathbb{N} - K\}. \quad (1)$$

Definition 1.4: A double sequence $x = (x_{ij}) \in {}_2\omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, we have

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: |x_{ij} - L| \geq \epsilon\} \in I. \quad (2)$$

In this case, we write $I - \lim x_{ij} = L$.

Definition 1.5: A double sequence $x = (x_{ij}) \in {}_2\omega$ is said to be I - null if L=0. In this case, we write

$$I - \lim x_{ij} = 0. \quad (3)$$

Definition 1.6: A double sequence $x = (x_{ij}) \in {}_2\omega$ is said to be I-Cauchy if for every $\epsilon > 0$ there exists numbers $m = m(\epsilon), n = n(\epsilon)$ such that

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: |x_{ij} - x_{mn}| \geq \epsilon\} \in I. \quad (4)$$

Definition 1.7: A double sequence $x = (x_{ij}) \in {}_2\omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{(i,j) \in \mathbb{N}\times\mathbb{N}: |x_{ij}| > M\} \in I. \quad (5)$$

Definition 1.8: A double sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies that $(\alpha_{ij}x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $(i,j) \in \mathbb{N}\times\mathbb{N}$.

Definition 1.9: A double sequence space E is said to be symmetric if $(x_{\pi(i,j)}) \in E$ whenever $(x_{ij}) \in E$, where $\pi(i,j)$ is a permutation on $\mathbb{N}\times\mathbb{N}$.

Definition 1.10: A double sequence space E is said to be sequence algebra if $(x_{ij}.y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 1.11: A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$, for all $(i,j) \in \mathbb{N}\times\mathbb{N}$.

Definition 1.12: Let $K = \{(n_i,k_j):(i,j): n_1 < n_2 < n_3 < \ldots. and\ k_1 < k_2 < k_3 < \ldots.\} \subseteq \mathbb{N}\times\mathbb{N}$ and E be a double sequence space. A K-step space of E is a sequence space

$$\lambda_K^E = \{(\alpha_{ij}x_{ij}): (x_{ij}) \in E\}.$$

where (α_{ij}) be double sequence of scalars.

Definition 1.13: A canonical preimage of a sequence $(a_{n_ik_j}) \in \lambda_K^E$ is a sequence $(b_{nk}) \in E$ defined as follows:

$$b_{nk} = \begin{cases} a_{nk}, & for\ n\ ,\ k\ \in K \\ 0, & otherwise\ . \end{cases}$$

Definition 1.14: A sequence space E is said to be monotone if it contains the canonical preimages of all its step-spaces.

Definition 1.15: Let $I = I_f$, the class of all finite subsets of $\mathbb{N}$. Then I is an admissible ideal in $\mathbb{N}$ and I_f convergence coincides with the usual convergence (Kostyrko et al., 2000, 2005).

Definition 1.16: Let X be a linear space. A function $g: X \to R$ is called paranorm, if for all $x, y \in X$,

1. $g(x) = 0\ if\ x = \theta$,
2. $g(-x) = g(x)$,
3. $g(x+y) \leq g(x) + g(y)$,
4. If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a(n \to \infty)$ in the sense that:

$g(x_n - a) \to 0(n \to \infty)$, then $g(\lambda_n x_n - \lambda a) \to 0(n \to \infty)$.

We used the following lemmas for establishing some results of this article.

Lemma 1.1: Every solid space is monotone (Tripathy and Hazarika, 2011).

Lemma 1.2: Let $K \in \mathcal{F}(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.

Lemma 1.3: If $I \subseteq 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

The following subspaces

$$l(p), l_\infty(p), c(p)\ and\ c_0(p)$$

where $p = (p_k)$ is a sequence of positive real numbers which were first introduced and discussed by Maddox (1969, 1986). After then, Lascarides (1971, 1983) defined the above sequence spaces in different manner.

The following inequalities (Khan and Tabassum, 2011) will be used throughout the paper. Let $p = (p_{ij})$ be a double sequence of positive real numbers.

For any complex λ, with $0 < p_{ij} \leq \sup_{ij} p_{ij} = H < \infty$, we have:

$$|\lambda|^{p_{ij}} \leq \max(1, |\lambda|^H).$$

Let $C = \max(1, 2^{H-1})$, then for the factorable sequences (a_{ij}) and (b_{ij}) in the complex plane, we have:

$$|a_{ij} + b_{ij}|^{p_{ij}} \leq C(|a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}}).$$

2. Main results

In this article, we introduce the following classes of double sequence spaces which are given as follows:

$$_2\mathcal{S}^I(p) = \{x = (x_{ij}) \in {}_2l_\infty : \{(i,j): |T(x_{ij}) - L|^{p_{ij}} \geq \epsilon\} \in I, for\ some\ L \in \mathbb{C}\} \quad (6)$$

$$_2\mathcal{S}_0^I(p) = \{x = (x_{ij}) \in {}_2l_\infty : \{(i,j): |T(x_{ij})|^{p_{ij}} \geq \epsilon\} \in I\}; \quad (7)$$

$$_2\mathcal{S}_\infty^I(p) = \{x = (x_{ij}) \in {}_2l_\infty : \exists K > 0\ s.t. \{(i,j): |T(x_{ij})|^{p_{ij}} \geq K\} \in I\}; \quad (8)$$

$$_2\mathcal{S}_\infty(p) = \{x = (x_{ij}) \in {}_2l_\infty : \sup_{ij}(|T(x_{ij})|)^{p_{ij}} < \infty\}. \quad (9)$$

We also denote,

$$_2\mathcal{M}_S^I(p) = {}_2\mathcal{S}^I(p) \cap {}_2\mathcal{S}_\infty(p)$$
$$_2\mathcal{M}_{S_0}^I(p) = {}_2\mathcal{S}_0^I(p) \cap {}_2\mathcal{S}_\infty(p).$$

where, $p = (p_{ij})$ is a bounded double sequence of positive real numbers.

Theorem 2.1: The classes of double sequences $_2\mathcal{S}_0^I(p)$, $_2\mathcal{S}^I(p)$, $_2\mathcal{M}_{S_0}^I(p)$, and $_2\mathcal{M}_S^I(p)$ are linear spaces.

Proof: Let $x = (x_{ij}), y = (y_{ij}) \in {}_2\mathcal{S}^I(p)$ be any two arbitrary elements and let α, β be scalars. Then, for a given $\epsilon > 0$, we have:

$$\{(i,j): |T(x_{ij}) - L_1|^{p_{ij}} \geq \frac{\epsilon}{2M_1}\}, for\ some\ L_1 \in \mathbb{C}\} \in I \quad (10)$$

and,

$$\{(i,j): |T(y_{ij}) - L_2|^{p_{ij}} \geq \frac{\epsilon}{2M_2}\}, for\ some\ L_2 \in \mathbb{C}\} \in I \quad (11)$$

where,

$$M_1 = D.\max\{1, \sup_{ij}|\alpha|^{p_{ij}}\}$$
$$M_2 = D.\max\{1, \sup_{ij}|\beta|^{p_{ij}}\}$$

and,

$$D = \max\{1, 2^{H-1}\}\ where\ H = \sup_{ij} p_{ij} \geq 0.$$

Let,

$$A_1 = \{(i,j): |T(x_{ij}) - L_1|^{p_{ij}} < \frac{\epsilon}{2M_1}, for\ some\ L_1 \in \mathbb{C}\} \in \mathcal{F}(I), \quad (12)$$

$$A_2 = \{(i,j): |T(y_{ij}) - L_2|^{p_{ij}} < \frac{\epsilon}{2M_2}, for\ some\ L_2 \in \mathbb{C}\} \in \mathcal{F}(I), \quad (13)$$

be such that $A_1^c, A_2^c \in I$.

Then,

$$A_3 = \{(i,j): |(\alpha T(x_{ij}) + \beta T(y_{ij})) - (\alpha L_1 + \beta L_2)|^{p_{ij}} < \epsilon\}$$
$$\supseteq [\{(i,j): |\alpha|^{p_{ij}}|T(x_{ij}) - L_1|^{p_{ij}} < \frac{\epsilon}{2M_1}|\alpha|^{p_{ij}}.D\}$$
$$\cap \{(i,j): |\beta|^{p_{ij}}|T(y_{ij}) - L_2|^{p_{ij}} < \frac{\epsilon}{2M_2}|\alpha|^{p_{ij}}.D\}],$$

implies that $A_3 \in \mathcal{F}(I)$. Thus $A_3^c = A_1^c \cup A_2^c \in I$. Therefore, $\alpha(x_{ij}) + \beta(y_{ij}) \in {}_2\mathcal{S}^I(p)$, for all scalars α, β and $(x_{ij}), (y_{ij}) \in {}_2\mathcal{S}^I(p)$. Hence $_2\mathcal{S}^I(p)$ is a linear space and the proof of others follow similarly.

Theorem 2.2: The classes of double sequences $_2\mathcal{M}_S^I(p)$ and $_2\mathcal{M}_S^I(p)$ are paranormed spaces, paranormed by,

$$g(x_{ij}) = \sup_{ij}|T(x_{ij})|^{\frac{p_{ij}}{M}}, \quad where\ M = \max\{1, \sup_{ij} p_{ij}\}.$$

Proof: Let $x = (x_{ij}), y = (y_{ij}) \in {}_2\mathcal{M}_S^I(p)$. (P_1) It is clear that $g(x) = 0$ if and only if $x = \theta$. (P_2) $g(-x) = g(x)$ is obvious. (P_3) Since $\frac{p_{ij}}{M} \leq 1$ and $M > 1$, using Minkowski's inequality, we have:

$$g(x_{ij} + y_{ij}) = \sup_{ij}|T(x_{ij} + y_{ij})|^{\frac{p_{ij}}{M}}$$
$$= \sup_{ij}|T(x_{ij}) + T(y_{ij})|^{\frac{p_{ij}}{M}}$$
$$\leq \sup_{ij}|T(x_{ij})|^{\frac{p_{ij}}{M}} + \sup_{ij}|T(y_{ij})|^{\frac{p_{ij}}{M}}$$
$$= g(x) + g(y).$$

Therefore,

$$g(x + y) \leq g(x) + g(y)$$

(P_4) Let (λ_{ij}) be a double sequence of scalars with $(\lambda_{ij}) \to \lambda(i,j \to \infty)$ and $(x_{ij}), L \in {}_2\mathcal{M}_S^I(p)$ such that,

$$x_{ij} \to L(i,j \to \infty),$$

in the sense that,

$$g(x_{ij} - L) \to 0(i,j \to \infty).$$

Then, since the inequality $g(x_{ij}) \leq g(x_{ij} - L) + g(L)$ holds by subadditivity of g, the sequence $g(x_{ij})$ is bounded. Therefore,

$$g[(\lambda_{ij}x_{ij} - \lambda L)] = g[(\lambda_{ij}x_{ij} - \lambda x_{ij} + \lambda x_{ij} - \lambda L)]$$
$$= g[(\lambda_{ij} - \lambda)x_{ij} + \lambda(x_{ij} - L)]$$
$$\leq g[(\lambda_{ij} - \lambda)x_{ij}] + g[\lambda(x_{ij} - L)]$$
$$\leq |(\lambda_{ij} - \lambda)|^{\frac{p_{ij}}{M}} g(x_{ij}) + |\lambda|^{\frac{p_{ij}}{M}} g(x_{ij} - L) \to 0,$$

as $(i,j \to \infty)$. That implies that the scalar multiplication is continuous. Hence $_2\mathcal{M}_S^I(p)$ is a paranormed space. For the another space $_2\mathcal{M}_{S_0}^I(p)$, the proof is similar.

Theorem 2.3: The set $_2\mathcal{M}_S^I(p)$ is closed subspace of $_2\mathcal{S}_\infty(p)$.

Proof: Let $(x_{ij}^{(pq)})$ be a Cauchy double sequence in ${}_2\mathcal{M}_S^I(p)$ such that $x^{(pq)} \to x$. We show that $x \in {}_2\mathcal{M}_S^I(p)$. Since $(x_{ij}^{(pq)}) \in {}_2\mathcal{M}_S^I(p)$, then there exists (a_{pq}), and for every $\epsilon > 0$ such that $\{(i,j): |x_{ij}^{(pq)} - a_{pq}|^{p_{ij}} \geq \epsilon\} \in I$. We need to show that:

1. (a_{pq}) converges to a, where a is some scalar.
2. If $U = \{(i,j): |T(x_{ij}) - a|^{p_{ij}} \leq \epsilon\}$, then $U^c \in I$.

Since $(x_{ij}^{(pq)})$ be a Cauchy double sequence in ${}_2\mathcal{M}_S^I(p)$ then for a given $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\sup_{ij} |T(x_{ij}^{(pq)}) - T(x_{ij}^{(rs)})| < \frac{\epsilon}{3}$, for all $p, q, r, s \geq k_0$. For a given $\epsilon > 0$, we have,

$$B_{pqrs} = \{(i,j): |T(x_{ij}^{(pq)}) - T(x_{ij}^{(rs)})|^{p_{ij}} < (\tfrac{\epsilon}{3})^M\},$$
$$B_{pq} = \{(i,j): |T(x_{ij}^{(pq)}) - a_{pq}|^{p_{ij}} < (\tfrac{\epsilon}{3})^M\},$$
$$B_{rs} = \{(i,j): |T(x_{ij}^{(rs)}) - a_{rs}|^{p_{ij}} < (\tfrac{\epsilon}{3})^M\}.$$

Then $B_{pqrs}^c, B_{pq}^c, B_{rs}^c \in I$. Let $B^c = B_{pqrs}^c \cap B_{pq}^c \cap B_{rs}^c$, where $B = \{(i,j): |a_{pq} - a_{rs}|^{p_{ij}} < \epsilon\}$, then $B^c \in I$. We choose $k_0 \in B^c$, then for each $p, q, r, s \geq k_0$, we have,

$$\{(i,j): |a_{pq} - a_{rs}|^{p_{ij}} < \epsilon\} \supseteq [\{(i,j): |T(x_{ij}^{(pq)}) - a_{pq}|^{p_{ij}} < (\tfrac{\epsilon}{3})^M\}$$
$$\cap \{(i,j): |T(x_{ij}^{(rs)}) - T(x_{ij}^{(pq)})|^{p_{ij}} < (\tfrac{\epsilon}{3})^M\}$$
$$\cap \{(i,j): |a_{rs} - T(x_{ij}^{(rs)})|^{p_{ij}} < (\tfrac{\epsilon}{3})^M\}].$$

Then (a_{pq}) is a cauchy double sequence in $\mathbb{C}$. So, there exists a scalar $a \in \mathbb{C}$ such that $(a_{pq}) \to a, as\ p, q \to \infty$.

For the next step, let $0 < \delta < 1$ be given. Then, we show that if:

$$U = \{(i,j): |T(x_{ij}^{(pq)}) - a|^{p_{ij}} \leq \delta\}$$

then $U^c \in I$. Since $x_{ij}^{(pq)} \to x$, then there exists $p_0, q_0 \in \mathbb{N}$ such that,

$$P = \{(i,j): |T(x_{ij}^{(p_0q_0)}) - T(x)|^{p_{ij}} < (\tfrac{\delta}{3D})^M\} \quad (14)$$

implies $P^c \in I$. The numbers p_0, q_0 can be so choosen that together with (14), we have:

$$Q = \{(i,j): |a_{p_0q_0} - a|^{p_{ij}} < (\tfrac{\delta}{3D})^M\}$$

such that $Q^c \in I$. Since $\left(x_{ij}^{(pq)}\right) \in {}_2\mathcal{M}_S^I(p)$. We have

$$\{(i,j): |T(x_{ij}^{(p_0q_0)}) - a_{p_0q_0}|^{p_{ij}} \geq \delta\} \in I.$$

Then we have a subset S of $\mathbb{N} \times \mathbb{N}$ such that $S^c \in I$, where

$$S = \{(i,j): |T(x_{ij}^{(p_0q_0)}) - a_{p_0q_0}|^{p_{ij}} < (\tfrac{\delta}{3D})^M\}.$$

Let $U^c = P^c \cup Q^c \cup S^c$, where,

$$U = \{(i,j): |T(x_{ij}) - a|^{p_{ij}} < \delta\}.$$

Therefore, for each $(i,j) \in U^c$, we have:

$$\{(i,j): |T(x) - a|^{p_{ij}} < \delta\} \supseteq [\{(i,j): |T(x_{ij}^{(p_0q_0)}) - T(x)|^{p_{ij}} < (\tfrac{\delta}{3})^M\}$$
$$\cap \{(i,j): |a_{p_0q_0} - a|^{p_{ij}} < (\tfrac{\delta}{3})^M\}$$
$$\cap \{(i,j): |T(x_{ij}^{(p_0q_0)}) - a_{p_0q_0}|^{p_{ij}} < (\tfrac{\delta}{3})^M\}].$$

Hence the result ${}_2\mathcal{M}_S^I(p) \subset {}_2\mathcal{S}_\infty(p)$ follows. Since the inclusions ${}_2\mathcal{M}_S^I(p) \subset {}_2\mathcal{S}_\infty(p)$and ${}_2\mathcal{M}_{S_0}^I(p) \subset {}_2\mathcal{S}_\infty(p)$are strict so in view of Theorem (2.3) we have the following result.

Theorem 2.4: The spaces ${}_2\mathcal{M}_S^I(p)$ and ${}_2\mathcal{M}_{S_0}^I(p)$ are nowhere dense subsets of ${}_2\mathcal{S}_\infty(p)$.

Theorem 2.5: The spaces ${}_2\mathcal{S}_0^I(p)$ and ${}_2\mathcal{M}_{S_0}^I(p)$ are both solid and monotone.

Proof: Here we consider ${}_2\mathcal{S}_0^I(p)$ and for ${}_2\mathcal{M}_{S_0}^I(p)$ the proof shall be similar. Let $x = (x_{ij}) \in {}_2\mathcal{S}_0^I(p)$ be an arbitrary element, then there exists $\epsilon > 0$ such that,

$$\{(i,j): |T(x_{ij})|^{p_{ij}} \geq \epsilon\} \in I \quad (15)$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$. Since $|\alpha|^{p_{ij}} \leq \max\{1, |\alpha|^G\} \leq 1$, for all $i, j \in \mathbb{N}$, where $G = \sup_{ij} p_{ij}$.

$$|T(\alpha_{ij}x_{ij})|^{p_{ij}} = |\alpha_{ij}T(x_{ij})|^{p_{ij}} \leq |T(x_{ij})|^{p_{ij}},\ for\ all\ i,j \in \mathbb{N}.$$
$$\{(i,j) \in \mathbb{N} \times \mathbb{N}: |T(\alpha_{ij}x_{ij})|^{p_{ij}} \geq \epsilon\} \subseteq \{(i,j) \in \mathbb{N} \times \mathbb{N}: |T(x_{ij})|^{p_{ij}} \geq \epsilon\}. \quad (16)$$

Thus we have $(\alpha_{ij}x_{ij}) \in {}_2\mathcal{S}_0^I(p)$. Hence ${}_2\mathcal{S}_0^I(p)$ is solid sequence space this shows that ${}_2\mathcal{S}_0^I(p)$ is monotone sequence space. Since every solid sequence space is monotone. For ${}_2\mathcal{M}_{S_0}^I(p)$ the proof shall be similar.

Theorem 2.6: For any Orlicz function M, the space ${}_2\mathcal{S}^I(p)$ and ${}_2\mathcal{M}_S^I(p)$ are neither solid nor monotone, if I is neither maximal nor $I = I_f$.

Proof: Here we give a counter example for establishment of this result. Let $X = {}_2\mathcal{S}^I$ and ${}_2\mathcal{M}_S^I$. Let us consider $I = I_\delta$.

Let $p_{ij} = 1, if\ k = i + j$ is even and $p_{ij} = 2, if\ k = i + j$ is odd . Consider, the K-step space $X_K(p)$ of $X(p)$ defined as follows. Let $x = (x_{ij}) \in X(p)$ and $y = (y_{ij}) \in X_K(p)$ be such that,

$$y_{ij} = \begin{cases} x_{ij}, & if\ k = i + j\ is\ even, \\ 0, & otherwise. \end{cases} \quad (17)$$

Consider the sequence (x_{ij}) defined by $(x_{ij}) = 1$ for all $i, j \in \mathbb{N}$. Then $x = (x_{ij}) \in {}_2\mathcal{S}^I(p)$ and ${}_2\mathcal{M}_S^I(p)$, but K-step space preimage does not belong to ${}_2\mathcal{S}^I(p)$ and ${}_2\mathcal{M}_S^I(p)$. Thus ${}_2\mathcal{S}^I(p)$ and ${}_2\mathcal{M}_S^I(p)$ are not monotone and hence they are not solid by Lemma (1.1).

Theorem 2.7: Let (p_{ij}) and (q_{ij}) be two double sequences of positive real numbers. Then ${}_2\mathcal{M}_{S_0}^I(p) \supseteq {}_2\mathcal{M}_{S_0}^I(q)$ if and only if $\liminf_{i,j\in K} \frac{p_{ij}}{q_{ij}} > 0$, where $K \subseteq \mathbb{N} \times \mathbb{N}$ such that $K \in \mathcal{F}(I)$.

Proof: Let $\liminf_{i,j\in K} \frac{p_{ij}}{q_{ij}} > 0$ and $(x_{ij}) \in {}_2\mathcal{M}_{S_0}^I(q)$. Then, there exists $\beta > 0$ such that $p_{ij} > \beta q_{ij}$ for sufficiently large $k \in K$. Since $(x_{ij}) \in {}_2\mathcal{M}_{S_0}^I(q)$. For a given $\epsilon > 0$, we have

$$B_0 = \{(i,j) \in \mathbb{N} \times \mathbb{N}: |T(x_{ij})|^{q_{ij}} \geq \epsilon\} \in I.$$

Let $G_0 = K^c \cup B_0$. Then for all sufficiently large $k \in G_0$.

$$\{(i,j) \in \mathbb{N} \times \mathbb{N}: |T(x_{ij})|^{p_{ij}} \geq \epsilon\} \subseteq \{(i,j) \in \mathbb{N} \times \mathbb{N}: |T(x_{ij})|^{\beta q_{ij}} \geq \epsilon\} \in I.$$

Therefore, $(x_{ij}) \in {}_2\mathcal{M}_{S_0}^I(q)$. The converse part of the result follows obviously.

Theorem 2.8: Let (p_{ij}) and (q_{ij}) be two double sequences of positive real numbers. Then ${}_2\mathcal{M}_{S_0}^I(q) \supseteq {}_2\mathcal{M}_{S_0}^I(p)$ if and only if $\liminf_{i,j\in K} \frac{q_{ij}}{p_{ij}} > 0$, where $K \subseteq \mathbb{N} \times \mathbb{N}$ such that $K \in \mathcal{F}(I)$.

Proof: The proof follows similarly as proof Theorem (2.7).

Theorem 2.9: Let (p_{ij}) and (q_{ij}) be two double sequences of positive real numbers.

Then ${}_2\mathcal{M}_{S_0}^I(q) = {}_2\mathcal{M}_{S_0}^I(p)$ if and only if

$$\liminf_{i,j\in K} \frac{p_{ij}}{q_{ij}} > 0$$

and

$$\liminf_{i,j\in K} \frac{q_{ij}}{p_{ij}} > 0,$$

where $K^c \subseteq \mathbb{N} \times \mathbb{N}$ such that $K \in I$.

Proof: By combining Theorem (2.7) and Theorem (2.8) we get the desired result.

3. Conclusion

In this paper, the notions of paranorm $I-$convergence of double sequence spaces defined by compact operator have been defined here and some elementary properties of these notions are obtained. These definitions and results provide new tools to deal with the convergence problems of sequences in the advance settings, occurring in many branches of science and engineering.

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Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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