

Fractional integral operator associated with extended Mittag-Leffler function



Aneela Nadir*, Adnan Khan

Department of Mathematics, National College of Business Administration and Economics, 40-E\1, Gullberg III, Lahore, Pakistan

ARTICLE INFO

Article history:

Received 23 July 2018

Received in revised form

8 December 2018

Accepted 12 December 2018

Keywords:

Laplace transform

Elzaki transform

Hadamard product

ABSTRACT

Popular literature of Special Functions includes the generalization and extensions of functions like gamma function, beta function, Mittag-Leffler function, hypergeometric function and confluent hypergeometric function etc. This sequel deals with the extension of Mittag-Leffler functions and its properties. We aim to find the composition of fractional integration formula known as \mathbb{P}_δ – transform with the extended Mittag-Leffler function. Some special cases and corollaries are pointed out which follow from our main results.

© 2019 The Authors. Published by IASE. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

1. Introduction

Fractional calculus is a subject, which got popularity during last four decades due to ability to deal with integrals and derivatives of any arbitrary real and complex order. It has demonstrated applications in diverse field of applied sciences and mathematics like diffusion, reaction-diffusion, fluid flow, polymer physics, chemical physics etc. A comprehensive account of fractional calculus operators and its applications and Special functions can be found in the monographs written by Agarwal and Choi (2016), Choi and Agarwal (2014, 2015), Gehlot (2013), Gupta and Parihar (2017), Kataria and Vellaisamy (2015), Kilbas et al. (2004), Nadir and Khan (2018a), Rahman et al. (2017a, 2017b), Rao et al. (2010), Saigo (1978), Saigo and Maeda (1998), Samko et al. (1993), Saxena and Parmar (2017), Shishkina and Sitnik (2017), Singh (2013), Srivastava and Agarwal (2013), Srivastava et al. (2012a, 2012b, 2017), Suthar et al. (2017), and references therein.

This paper deals with operator of fractional integration known as \mathbb{P}_δ – transform and Mittag-Leffler function is considered for numerical computation. For the last few decades, many authors are interested in in generalization of this function. In this continuation, the work of Desai et al. (2016), Kilbas et al. (2004), Mittal et al. (2016), Nadir et al.

(2014), Nadir and Khan (2018b), Özarlan and Yilmaz (2014), Parmar (2015), Prabhakar (1971), Srivastava and Tomovski (2009), and cited references therein can be consulted.

For the present sequel, we consider the following definitions and the related work.

Definition: The extension of Mittag-Leffler function is defined and studied in the following way (Parmar, 2015; Nadir and Khan, 2018b):

$$E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in N_0}; \gamma)}(z; p) = \sum_{k=0}^{\infty} \frac{B_p^{(\{\kappa_n\}_{n \in N_0})}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{z^k}{\Gamma(\xi k + \zeta)} \quad (1)$$

$(z, \xi, \gamma \in \mathbb{C}; \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\gamma) > 1; p \geq 0)$

where $B_p^{(\{\kappa_n\}_{n \in N_0}; \gamma)}$ is the extension of a Beta function corresponding to the function $\Theta(\{\kappa_n\}_{n \in N_0}; z)$ defined by Srivastava et al. (2012b) in the following manners:

$$\Theta(\{\kappa_n\}_{n \in N_0}; z) := \sum_{n=0}^{\infty} \kappa_n \frac{z^n}{n!} \begin{pmatrix} |z| < \Re \\ 0 < \Re < \infty \\ \kappa_0 := 1 \end{pmatrix} \left(\begin{matrix} \Re(z) \rightarrow \infty \\ \Re_0 > 0; \omega \in \mathbb{C} \end{matrix} \right) \quad (2)$$

$(\mathfrak{m}_0 z^{\omega} \exp(z) [1 + O(\frac{1}{z})])$

Here, it is noted that function defined by Parmar (2015) is a generalization of all extensions of [ML-] Mittag-Leffler function defined in literature, depending upon the value of a bounded sequence κ_n and upon parameters ξ, ζ, γ, p . Some cases are listed below.

When $\kappa_n = 1, (n \in \mathbb{N})$ the function defined in (1) becomes function considered by Özarlan and Yilmaz (2014).

* Corresponding Author.

Email Address: aneelanadir@yahoo.com (A. Nadir)<https://doi.org/10.21833/ijaas.2019.02.001>

Corresponding author's ORCID profile:

<https://orcid.org/0000-0003-2475-1661>

2313-626X/© 2019 The Authors. Published by IASE.

This is an open access article under the CC BY-NC-ND license

[\(http://creativecommons.org/licenses/by-nc-nd/4.0/\)](http://creativecommons.org/licenses/by-nc-nd/4.0/)

$$E_{\xi, \zeta}^{\gamma}(z; p) = \sum_{k=0}^{\infty} \frac{B(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{z^k}{\Gamma(\xi k + \zeta)} \quad (3)$$

$$\left(\begin{array}{l} z, \zeta, \gamma \in \mathbb{C}; \Re(\xi) > 0, \\ \Re(\zeta) > 0, \Re(\gamma) > 1; p \geq 0 \end{array} \right)$$

In particular when $\kappa_n = 0$, (1) yields the Mittag-Leffler function due to Prabhakar (1971):

$$E_{\xi, \zeta}^{\gamma}(z; p) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\xi k + \zeta) k!} \quad (4)$$

$$(z, \xi, \zeta, \gamma \in \mathbb{C}; \Re(\xi) > 0, \Re(\zeta) > 0)$$

when we set the value of $\kappa_n = \frac{(\rho)_n}{(\sigma)_n}, (n \in \mathbb{N}_0)$, then extension will be

$$E_{\xi, \zeta}^{(\rho, \sigma); \gamma}(z; p) = \sum_{k=0}^{\infty} \frac{B^{(\rho, \sigma)}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{z^k}{\Gamma(\xi k + \zeta)} \quad (5)$$

$$\left(\begin{array}{l} z, \zeta, \gamma \in \mathbb{C}; \Re(\rho) > 0, \Re(\sigma) > 0 \\ \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\gamma) > 1; p \geq 0 \end{array} \right)$$

Further special cases of [ML] Mittag-Leffler function defined by Parmar (2015) by the selection of $\xi = \zeta = 1$, the extensions defined in (3), (4) and (5) reduces to the confluent hypergeometric function as

$$E_{1,1}^{(\{\kappa_n\}_{n \in \mathbb{N}_0}; \gamma)}(z; p) = \Phi_p^{(\{\kappa_n\}_{n \in \mathbb{N}_0})}(\gamma; 1; z) \quad (6)$$

$$E_{1,1}^{(p, q); \gamma}(z; p) = \Phi_p^{(p, q)}(\gamma; 1; z) \quad (7)$$

$$E_{1,1}^{\gamma}(z; p) = \Phi_p(\gamma; 1; z). \quad (8)$$

In order to establish the main results, we need the concept of Hadamard Product.

Definition: Let $g(z) := \sum_{k=0}^{\infty} x_k z^k$ and $h(z) := \sum_{k=0}^{\infty} y_k z^k$ be two power series then the Hadamard Product of the two series is defined as (Pohlen, 2009):

$$(g * h)(z) := \sum_{k=0}^{\infty} x_k y_k z^k = (h \cdot g)(z) \quad (9)$$

$$(|z| < R)$$

where

$$R = \lim_{k \rightarrow \infty} \left| \frac{x_k y_k}{x_{k+1} y_{k+1}} \right| = \left(\lim_{k \rightarrow \infty} \left| \frac{x_k}{x_{k+1}} \right| \right) \cdot \left(\lim_{k \rightarrow \infty} \left| \frac{y_k}{y_{k+1}} \right| \right) = R_g \cdot R_h. \quad (10)$$

Where R_g and R_h are the radii of convergence of two series $g(z)$ and $h(z)$ respectively. Therefore, in general, $R \geq R_g \cdot R_h$.

It is to be noted that if one of the power series is an analytical function, then the Hadamard product series is also an analytical function.

Definition: The \mathbb{P}_δ -transform $\mathbb{P}_\delta[f(t); s]$ of a function $f(t)$ where s is a complex variable defined by (Kumer, 2013):

$$\mathbb{P}_\delta[f(t); s] = F_{\mathbb{P}_\delta}(s) := \int_0^\infty [1 + (\delta - 1)s]^{-\frac{t}{\delta-1}} f(t) dt, \quad (11)$$

$$(\delta > 1)$$

provided that the sufficient existence conditions given by Lemma below be satisfied.

Lemma: Let the function $f(t)$ be integrable over any finite interval $(a, b), (0 < a < t < b)$; suppose also that there exists a real number c such that each of the following assertions holds true:

- (i) For any arbitrary $b > 0, \int_b^\infty e^{-ct} f(t) dt$ tends to a finite limit as $\rho \rightarrow \infty; \zeta \rightarrow \infty$
- (ii) For any arbitrary $a > 0, \int_0^a |f(t)| dt$ tends to finite limit as $\rho \rightarrow 0 +$.

Then the \mathbb{P}_δ -transform $\mathbb{P}_\delta[f(t); s]$ exists whenever

$$\Re\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right) > c \quad (s \in \mathbb{C})$$

The power function of the transform is given below.

$$\mathbb{P}_\delta[t^{\beta-1}; s] := \left(\frac{\delta-1}{\ln[1+(\delta-1)s]}\right)^\beta \Gamma(\beta), \quad (12)$$

$$(\Re(\beta) > 0; \delta > 1)$$

It is to be noted that when transform is converted into classical Laplace transform. The integral form of a classical Laplace transform is given below, for reference see Sneddon (1979).

$$\mathcal{L}[f(t); s] := \int_0^\infty e^{-st} f(t) dt \quad (13)$$

Elzaki Transform a new integral transform is a modification of Laplace Transform and is defined as

Definition: Consider a function of exponential order is defined in a set as:

$$W = \left\{ \begin{array}{l} f(t): \exists M, s_1, s_2 > 0, |f(t)| < M e^{s_1 t}, \text{ if } t \in (-1)^j \times \\ [1, \infty) \end{array} \right\} \quad (14)$$

where M, s_1 and s_2 are constant. M must be finite and s_1, s_2 may or may not be finite. Elzaki Transform for a function $f(t)$ is defined as

$$E[f(t)] := T(c) = c \int_0^\infty e^{-\frac{t}{c}} f(t) dt, t \geq 0 \quad (15)$$

where the variable c is used to factorize the variable t in the argument of the function f .

2. \mathbb{P}_δ -transform of extended Mittag-Leffler function

In this section, we consider the composition of \mathbb{P}_δ - integral transform of pathway type with extended Mittag-Leffler function $E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in \mathbb{N}_0}; \gamma)}(z; p)$. Some interesting special cases are discussed and the results are obtained in the form of two analytical functions with the help of Hadamard product of $E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in \mathbb{N}_0}; \gamma)}(z; p)$ and Gauss hypergeometric function. Let us start our main theorem.

Theorem: Let $z, \zeta, \gamma \in \mathbb{C}; \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\gamma) > 1; p \geq 0$, then the \mathbb{P}_δ -transform holds and exhibit the following relation.

$$\mathbb{P}_\delta \left[t^{\sigma-1} E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in N_0}; \gamma)}(zt; p); s \right] = \frac{\Gamma(\sigma)}{[\mathfrak{N}(\delta; s)]^\sigma} E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in N_0}; \gamma)} \left(\frac{z}{[\mathfrak{N}(\delta; s)]}; p \right) * F_0^1 \left[\begin{matrix} \sigma; \\ -; \end{matrix} \frac{z}{[\mathfrak{N}(\delta; s)]} \right] \quad (16)$$

($|z| < 1; \min\{\Re(s), \Re(\sigma)\} > 0; \delta > 1$)

where

$$[\mathfrak{N}(\delta; s)] := \frac{\ln[1+(\delta-1)s]}{\delta-1} \quad (17)$$

where F_0^1 is a well known Gauss hypergeometric function (Rainville, 1971).

Proof: For the sake of convenience, let us denote the left hand side by Ω , we have

$$\Omega = \mathbb{P}_\delta \left[t^{\sigma-1} E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in N_0}; \gamma)}(zt; p); s \right] = \int_0^\infty t^{\sigma-1} [1 + (\delta-1)s]^{-\frac{t}{\delta-1}} E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in N_0}; \gamma)}(zt; p) dt = \int_0^\infty t^{\sigma-1} [1 + (\delta-1)s]^{-\frac{t}{\delta-1}} \quad (18)$$

$$\sum_{k=0}^\infty \frac{B_p^{(\{\kappa_n\}_{n \in N_0})}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{(zt)^k}{\Gamma(\xi k + \zeta)} dt. \quad (19)$$

Due to uniform convergence, order of integration and summation can be changed and then replacing the variable β by $\sigma + k$ in (12), we get

$$\Omega = \sum_{k=0}^\infty \frac{B_p^{(\{\kappa_n\}_{n \in N_0})}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \quad (20)$$

$$\frac{(z)^k}{\Gamma(\xi k + \zeta)} \int_0^\infty t^{\sigma+k-1} [1 + (\delta-1)s]^{-\frac{t}{\delta-1}} dt = \sum_{k=0}^\infty \frac{B_p^{(\{\kappa_n\}_{n \in N_0})}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{(z)^k}{\Gamma(\xi k + \zeta)} \frac{\Gamma(\sigma+k)}{[\mathfrak{N}(\delta; s)]^{\sigma+k}} =$$

$$\frac{\Gamma(\sigma)}{[\mathfrak{N}(\delta; s)]^\sigma} \sum_{k=0}^\infty \frac{B_p^{(\{\kappa_n\}_{n \in N_0})}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \quad (21)$$

$$\left(\frac{z}{[\mathfrak{N}(\delta; s)]} \right)^k \frac{\Gamma(\sigma+k)}{\Gamma(\xi k + \zeta) \Gamma(\sigma)}. \quad (22)$$

Thus, the last expression can easily be obtained by means of Hadamard Product as in (9).

$$\Omega = \frac{\Gamma(\sigma)}{[\mathfrak{N}(\delta; s)]^\sigma} E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in N_0}; \gamma)} \left(\frac{z}{[\mathfrak{N}(\delta; s)]}; p \right) * F_0^1 \left[\begin{matrix} \sigma; \\ -; \end{matrix} \frac{z}{[\mathfrak{N}(\delta; s)]} \right] \quad (23)$$

this completes the proof.

Different extensions of Mittag-Leffler function defined in literature are the special cases of our main result. Further, these extensions depend upon the values of the parameters $\xi, \zeta, \gamma \in \mathbb{C}$ and the bounded sequence κ_n . Here, we discuss some examples, which are connected to our main result.

If we select the sequence $\kappa_n = 1, (n \in N_0)$ then the proposed function reduces to the definition of Özarслан and Yilmaz (2014).

Corollary 1: Let $z, \zeta, \gamma \in \mathbb{C}$ be such that

$$\Re(\xi) > 0, \Re(\zeta) > 0, \Re(\gamma) > 0; p \geq 0$$

then we have the following relation

$$\mathbb{P}_\delta \left[t^{\sigma-1} E_{\xi, \zeta}^\gamma(zt; p); s \right] = \frac{\Gamma(\sigma)}{[\mathfrak{N}(\delta; s)]^\sigma} E_{\xi, \zeta}^\gamma \left(\frac{z}{[\mathfrak{N}(\delta; s)]}; p \right) * F_0^1 \left[\begin{matrix} \sigma; \\ -; \end{matrix} \frac{z}{[\mathfrak{N}(\delta; s)]} \right]. \quad (24)$$

If we set the value of the parameters, $\xi = \zeta = 1$ the extension of the proposed function takes the form of extended confluent hypergeometric function then the main theorem reduces to the following corollary.

Corollary 2: Let $z, \zeta, \gamma \in \mathbb{C}$ then the following relation holds true.

$$\mathbb{P}_\delta \left[t^{\sigma-1} \phi_p^{(\{\kappa_n\}_{n \in N_0})}(\gamma; 1; z); s \right] = \frac{\Gamma(\sigma)}{[\mathfrak{N}(\delta; s)]^\sigma} \phi_p^{(\{\kappa_n\}_{n \in N_0})} \left(\gamma; 1; \frac{z}{[\mathfrak{N}(\delta; s)]} \right) * F_0^1 \left[\begin{matrix} \sigma; \\ -; \end{matrix} \frac{z}{[\mathfrak{N}(\delta; s)]} \right]. \quad (25)$$

Remark: setting $\kappa_n = \frac{(\rho)_n}{(\sigma)_n}, (n \in N_0)$, then results of our main theorem reduces for the result of the function.

$$E_{\xi, \zeta}^{(\rho, \sigma); \gamma}(z; p) = \sum_{k=0}^\infty \frac{B^{(\rho, \sigma)}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{z^k}{\Gamma(\xi k + \zeta)}$$

Further, for setting $p = 0$ then function will become Prabhakar's definition and we get the results as in Kumar (2013). Setting $\xi = \zeta = 1$ and the bounded sequences $\kappa_n = 1$ then the expression yields the result for extended confluent hypergeometric function defined by Chaudhry et al. (2004). Further, for setting $p = 0$ and $\xi = \zeta = 1$ then the result is for the classical confluent hypergeometric function.

3. Further special cases and concluding remarks

It is to be noted that \mathbb{P}_δ -transform reduces to classical Laplace transform by converting the variable $\frac{\ln[1+(\delta-1)s]}{\delta-1} \rightarrow s$ then we get integral involving Laplace transform stated in corollaries.

Corollary 3: Let $z, \zeta, \gamma \in \mathbb{C}; \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\gamma) > 1; p \geq 0$ then Laplace Transform formula holds true and establish the following result.

$$\mathcal{L} \left[t^{\sigma-1} E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in N_0}; \gamma)}(zt; p); s \right] = \frac{\Gamma(\sigma)}{s^\sigma} E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in N_0}; \gamma)} \left(\frac{z}{s}; p \right) * F_0^1 \left[\begin{matrix} \sigma; \\ -; \end{matrix} \frac{z}{s} \right] \quad (26)$$

$\Re(p) > 0, \Re(s) > 0, \Re(\xi) > 0, \Re(\zeta) > 0$

Proof:

$$\mathcal{L} \left[t^{\sigma-1} E_{\xi, \zeta}^{(\{\kappa_n\}_{n \in N_0}; \gamma)}(zt; p); s \right] = \int_0^\infty t^{\sigma-1} e^{-st} \left(\sum_{k=0}^\infty \frac{B_p^{(\{\kappa_n\}_{n \in N_0})}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{(zt)^k}{\Gamma(\xi k + \zeta)} \right) dt \quad (27)$$

$$= \left(\sum_{k=0}^\infty \frac{B_p^{(\{\kappa_n\}_{n \in N_0})}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{(z)^k}{\Gamma(\xi k + \zeta)} \right) \int_0^\infty t^{\sigma+k-1} e^{-st} dt \quad (28)$$

$$= \sum_{k=0}^{\infty} \frac{E_p^{(\{k_n\}_{n \in N_0})}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{(z)^k}{\Gamma(\xi k + \zeta)} \frac{\Gamma(\sigma+k)}{s^{\sigma+k}} \quad (29)$$

Thus, it is to be noted that we get the same result through \mathbb{P}_δ –transform by changing the variable $\frac{\ln[1+(\delta-1)s]}{\delta-1}$ into s .

Corollary 4: Laplace Transform formula holds true

$$\begin{aligned} &\mathcal{L} \left[t^{\sigma-1} E_{\xi, \zeta}^{\gamma}(zt; p); s \right] \\ &= \frac{\Gamma(\sigma)}{s^{\sigma}} E_{\xi, \zeta}^{\gamma} \left(\frac{z}{s}; p \right) * F_0^1 \left[\begin{matrix} \sigma; z \\ -; s \end{matrix} \right] \end{aligned} \quad (30)$$

$\Re(p) > 0, \Re(s) > 0, \Re(\xi) > 0, \Re(\zeta) > 0.$

Corollary 5: Laplace Transform formula holds true

$$\begin{aligned} &\mathcal{L} \left[t^{\sigma-1} E_{\xi, \zeta}^{(\rho, \sigma; \gamma)}(zt; p); s \right] \\ &= \frac{\Gamma(\sigma)}{s^{\sigma}} E_{\xi, \zeta}^{(\rho, \sigma; \gamma)} \left(\frac{z}{s}; p \right) * F_0^1 \left[\begin{matrix} \sigma; z \\ -; s \end{matrix} \right] \end{aligned} \quad (31)$$

$\Re(p) > 0, \Re(s) > 0, \Re(\xi) > 0, \Re(\zeta) > 0$

Remark: In particular, the \mathbb{P}_δ –transform as in (11) reduces immediately to the Elzaki Transform by the following change of variables.

$$c \rightarrow \frac{\delta-1}{\ln[1+(\delta-1)s]} \quad (32)$$

Hence, it is easy to see that two integrals have the following relationship.

$$\mathbb{P}_\delta = [\mathfrak{N}(\delta; s)]E(f(t)) \quad (33)$$

4. Conclusion

It is noted that the extended Mittag-Leffler function defined by Parmar (2015) is more general in nature and various generalized types of Mittag-Leffler function and confluent hypergeometric functions defined in literature can easily be derived through the extended form. Similarly, the \mathbb{P}_δ –transform defined by Kumar (2013) (fractional integral operator) enable us to convert the table of Laplace transform and the Elzaki transform into the corresponding transform and vice versa.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

References

Agarwal P and Choi J (2016). Fractional calculus operators and their image formulas. Journal of the Korean Mathematical Society, 53(5): 1183-1210. <https://doi.org/10.4134/JKMS.j150458>

Chaudhry MA, Qadir A, Srivastava HM, and Paris RB (2004). Extended hypergeometric and confluent hypergeometric functions. Applied Mathematics and Computation, 159(2): 589-602. <https://doi.org/10.1016/j.amc.2003.09.017>

Choi J and Agarwal P (2014). Certain integral transform and fractional integral formulas for the generalized Gauss hypergeometric functions. Abstract and Applied Analysis, 2014: Article ID 735946. <https://doi.org/10.1155/2014/735946>

Choi J and Agarwal P (2015). Certain integral transforms for the incomplete functions. Applied Mathematics and Information Sciences, 9(4): 2161-2167.

Desai R, Salehbbhai IA, and Shukla AK (2016). Note on generalized Mittag-Leffler function. SpringerPlus, 5(1): 683-691. <https://doi.org/10.1186/s40064-016-2299-x>

Gehlot KS (2013). Integral representation and certain properties of m-series associated with fractional calculus. In International Mathematical Forum, 8(9): 415-426. <https://doi.org/10.12988/imf.2013.13041>

Gupta A and Parihar CL (2017). Saigo’s k –fractional calculus operators. Malaya Journal of Matematik, 5(3): 494-504.

Kataria KK and Vellaisamy P (2015). The generalized K-Wright function and marichev-saigo-maeda fractional operators. Journal of Analysis, 23: 75-87.

Kilbas AA, Saigo M, and Saxena RK (2004). Generalized Mittag-Leffler function and generalized fractional calculus operators. Integral Transforms and Special Functions, 15(1): 31-49. <https://doi.org/10.1080/10652460310001600717>

Kumar D (2013). Solution of fractional kinetic equation by a class of integral transform of pathway type. Journal of Mathematical Physics, 54(4): 043509. <https://doi.org/10.1063/1.4800768>

Mittal E, Pandey RM, and Joshi S (2016). On extension of Mittag-Leffler function. Applications and Applied Mathematics, 11(1): 307-316.

Nadir A and Khan A (2018a). Marichev-Saigo-Maeda differential operator and generalized incomplete hypergeometric functions. Journal of Mathematics, 50(3): 123-129.

Nadir A and Khan A (2018b). Caputo MSM fractional differentiation of extended Mittag-Leffler function. International Journal of Advanced and Applied Sciences. 5(10): 28-34. <https://doi.org/10.21833/ijaas.2018.10.005>

Nadir A, Khan A, and Kalim M (2014). Integral transforms of the generalized mittag-leffler function. Applied Mathematical Sciences, 8(103): 5145-5154. <https://doi.org/10.12988/ams.2014.43218>

Özarslan MA and Yilmaz B (2014). The extended Mittag-Leffler function and its properties. Journal of Inequalities and Applications, 2014(1): 85-95. <https://doi.org/10.1186/1029-242X-2014-85>

Parmar RK (2015). A Class of extended mittag-leffler functions and their properties related to integral transforms and fractional calculus. Mathematics, 3(4): 1069-1082. <https://doi.org/10.3390/math3041069>

Pohlen T (2009). The Hadamard product and universal power series. Ph.D. Dissertation, University in Trier, Trier, Germany.

Prabhakar TR (1971). A singular integral equation with a generalized Mittag Leffler function in the kernel. Yokohama Mathematical Journal, 19: 7-15.

Rahman G, Baleanu D, Qurashi MA, Purohit SD, Mubeen S and Arshad M (2017a). The extended Mittag-Leffler function via fractional calculus. Journal of Nonlinear Sciences and Applications. 10: 4244-4253. <https://doi.org/10.22436/jnsa.010.08.19>

Rahman G, Ghaffar A, Mubeen S, Arshad M, and Khan SU (2017b). The composition of extended Mittag-Leffler functions with pathway integral operator. Advances in Difference Equations, 2017(176): 1-10. <https://doi.org/10.1186/s13662-017-1237-8>

- Rainville ED (1971). Special functions. 2nd Edition, Chelsea Publishing Company, Bronx, New York, USA.
- Rao A, Garg M, and Kalla SL (2010). Caputo-type fractional derivative of a hypergeometric integral operator. Kuwait Journal of Science and Engineering, 37(1A): 15-29.
- Saigo M (1978). A remark on integral operators involving the Gauss hypergeometric functions. Kyushu University, 11: 135-143.
- Saigo M and Maeda N (1998). More generalization of fractional calculus. In: Rusev P, Dimovski I, and Kiryakova V (Eds.), Transform methods and special functions: 386-400. Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Varna, Bulgaria. **PMCID:PMC106901**
- Samko SG, Kilbas AA, and Marichev OI (1993). Fractional integrals and derivatives: Theory and applications. Gordon and Breach, Yverdon, Switzerland.
- Saxena RK and Parmar RK (2017). Fractional integration and differentiation of the generalized mathieu series. Axioms, 6(3): 18-29.
<https://doi.org/10.3390/axioms6030018>
- Shishkina EL and Sitnik SM (2017). On fractional powers of Bessel operators. Journal of Inequalities and Special Functions, 8(1): 49-67.
- Singh DK (2013). On extended M-series. Malaya Journal of Matematik, 1(1): 57-69.
- Sneddon IN (1979). The use of the integral transform. Tata McGraw Hill, New Delhi, India.
- Srivastava HM and Agarwal P (2013). Certain fractional integral operators and the generalized incomplete hypergeometric functions. Applications and Applied Mathematics, 8(2): 333-345.
- Srivastava HM and Tomovski Ž (2009). Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. Applied Mathematics and Computation, 211(1): 198-210.
<https://doi.org/10.1016/j.amc.2009.01.055>
- Srivastava HM, Chaudhry MA, and Agarwal RP (2012a). The incomplete Pochhammer symbols and their applications to hypergeometric and related functions. Integral Transforms and Special Functions, 23(9): 659-683.
<https://doi.org/10.1080/10652469.2011.623350>
- Srivastava HM, Parmar RK, and Chopra P (2012b). A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions. Axioms, 1(3): 238-258.
<https://doi.org/10.3390/axioms1030238>
- Srivastava R, Agarwal R, and Jain S (2017). A family of the incomplete hypergeometric functions and associated integral transform and fractional derivative formulas. Filomat, 31(1): 125-140.
<https://doi.org/10.2298/FIL1701125S>
- Suthar DL, Parmar RK, and Purohit SD (2017). Fractional calculus with complex order and generalized hypergeometric functions. Nonlinear Science Letters A, 8: 156-161.