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Quantile mechanics: Issues arising from critical review

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ABSTRACT

Approximations are the alternative way of obtaining the Quantile function when the inversion method cannot be applied to distributions whose cumulative distribution functions do not have close form expressions. Approximations come in form of functional approximation, numerical algorithm, closed form expressed in terms of others and series expansions. Several quantile approximations are available which have been proven to be precise, but some issues like the presence of shape parameters, inapplicability of existing methods to complex distributions and low computational speed and accuracy place undue limitations to their effective use. Quantile mechanics (QM) is a series expansion method that addressed these issues as evidenced in the paper. Quantile mechanics is a generalization of the use of ordinary differential equations (ODE) in quantile approximation. The paper is a review that critically examined with evidences; the formulation, applications and advantages of QM over other surveyed methods. Some issues bothering on the use of QM were also discussed. The review concluded with areas of further studies which are open for scientific investigation and exploration.

1. Introduction

In probability, statistics and stochastic analysis, quantile function (QF) is one of the ways of characterizing probability distributions. Probability distributions can be discrete, continuous or mixed. Other probability functions are probability density function (PDF), cumulative distribution function (CDF), survival function, inverse survival function, hazard function, odd function and reversed hazard function.

Quantile function is the inverse cumulative distribution. The details of the mathematical formulation, theories, description, estimation and properties can be seen in the works of Parzen (2004) and the references therein. The QF is unique irrespective of the mode orientation of the distribution (unimodal, bimodal or multimodal). Research on the QF is fueled by their wide applicability in modeling real life phenomena. This is extended to simulation of physical and other related systems. This assumes the form of random number generation, Monte Carlo simulation, copulas and other uses.

The inversion method is the simplest method of obtaining the QF from CDF. This is however limited to the case where the CDF and QF have closed form expressions. Some important distributions do not have closed form representations. Some of the distributions are: normal, beta, Erlang, MacDonald, chi, Lévy, hyperbolic, beta prime, chi-square, gamma, student's t, F distribution and others.

The alternative route followed by researchers when the inversion method cannot be applied are approximations which may come in the form of series expansions, closed form or functional approximation, numerical algorithm and the closed form expression drafted in terms of the quantile function of another distribution.

Several rational and closed form approximations have been obtained for some distributions which are shown in Table 1.

Despite the progress made in finding analytic expression and approximation of the quantile functions of probability distributions as shown in Table 1, several issues bothering on the methods of quantile approximations are yet to be discussed. The issues arose from the inability of the age-long "inversion method" to estimate the quantile function

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of some probability distributions with intractable cumulative distribution functions (CDF). Secondly, the presence of shape parameters in distributions like the Chi-square, gamma, Erlang, Beta makes approximation of quantile functions very tedious and challenging. Thirdly, all those methods of the approximations are fast, efficient but failed when applied to complex distributions and lastly, the available methods have issues bothered on convergence and precision.

Table 1: Survey of contributions to quantile approximation

Distribution	Details
Chi-square	Lin (1994), Ittrich et al. (2000).
Beta	Thomson (1947), Gil et al. (2017).
Behrens-Fisher	Patil (1969), Davis and Scott (1973).
F	Abernathy and Smith (1993a, b), Torigoe (2011).
Gamma	Withers and Nadarajah (2014), Gil et al. (2015).
Nakagami	Bilim and Develi (2015), Kabalci (2018).
Non-central chi-square	Torigoe (1996), Sahai and Ojeda (2009).
Non-central student t	Akahira (1995), Sahai and Ojeda (1998).
Normal (probit)	Odeh and Evans (1974), Beasley and Springer (1977), Wichura (1987, 1988), Lin (1990), Acklam (2003), Soranzo and Epure (2014).
Pearson(general)	Bowman and Shenton (1979).
Student t	Shaw (2006), Schluter and Fischer (2012).

This review is to survey how quantile mechanics (QM) has addressed the aforementioned issues. Also the advantages, limitations of the QM and areas of further studies are included. Quantile Mechanics (QM) introduced by Steinbrecher and Shaw (2008) is relatively a recent series expansion method of quantile approximation. The approach is a departure from the use of CDF to the use of the derivative of the reciprocal of the PDF of the given distribution in the estimation of the quantile function. The outcome is a second order nonlinear ordinary differential equation (ODE) whose solution, in series form determines the quantile function of the given distribution. The solution of the ODE has been often in the form of Taylor series, power series and asymptotic expansions of the given quantile function. The availability of those series and expansion can present interesting possibilities in the applications of the quantile function. The details on the applications of the quantile functions are included in the paper. Evidences from different contributions to the use of QM showed that the approach is a major improvement over other methods in terms of efficiency and quantile approximation of complex distributions. Quantile mechanics can also be extended to distributions with known characteristic functions, but unknown or partially known PDF and CDF. Quantile mechanics can be applied to the quantile approximation of multivariate probability distributions. Recently, the concept of the QM has been applied to the development of efficient and robust algorithms for quantile approximations. The algorithms are found to be an appreciable improvement over the existing ones. Quantile mechanics are the formalization and generalization of using the ODE to approximate the QF of probability distributions.

In general, before the advent of QM, a look at the earlier research done on the ODE of probability functions showed that the emphasis was on models and not on probability distributions. The use of ODE comes in form of the following:

a) Modeling of probability of events, Probability density function (PDF) and cumulative distribution function (CDF) of distributions; (Xiao et al., 2015) amongst other authors.
b) Modeling of CDF of probabilistic or random models (Lee and Taubman, 2006).
c) Modeling of PDF of probabilistic models (Popkov and Dubnov, 2016).
d) Modeling of hazard function (HF) and survival function (SF) of probabilistic models (Wang et al., 2016).
e) Modeling of other aspects of probabilistic models for example, the moment, generating functions and so on (Csenki, 2015).

The few works on the use of ODE for QF are the key foundation of the QM.

However, some gaps remain unfilled in the utilization of the QM approach. Now that the QM approach has proved to be an efficient method as verified by the researchers, it remains to be seen if other similar distributions to the uniform distribution like u-quadratic, triangular, rectangular and trapezoidal distributions can be used in lieu of the uniform distribution, especially in random number generation where it is used. Quantile mechanics is plagued with inconsistencies in the solution of the second order nonlinear ODE obtained as the output of the approach. Different series solution methods can be used with different levels of comparative results. It remains to be shown what the derivative of the reciprocal of different probability functions will look like since the QM is based on the derivative of the reciprocal of the PDF of the given distribution. Quantile mechanics is a very complex method for simple distributions. This is due to the procedure of the QM that depends on the creating and finding a definite solution to the nonlinear ODE that characterize the given distribution.

In order to fully understand the progress made in the creation of the QM, there is a need to compare and measure the solutions (approximation) obtained through the first and second order nonlinear ODEs obtained via the QM. Further comparison may come in the way of assessing the QM with the ODE obtained through the simple differentiation of the quantile function of the given distribution. This of course can be possible if the QF of the distribution is differentiable. Inconsistencies in the initial or boundary conditions that define the nonlinear ODE obtained through ODE has been noticed and noted. This amongst others has created difficulty in the extension of the QM approach to target distributions, for example, in finding the base distribution needed to find the approximation of the base distribution. Moreover, the QM and other methods are yet to be applied in the quantile approximation of convoluted, ratio and product probability distributions. These mentioned distributions are very important in the statistical modeling. In the same vein, the search continues for the method that can be used in the quantile approximation of discrete distribution. This is because the nature of the discrete distributions is incompatible with QM approach and as such the QM joins the queue of methods that are helpless in the quantile approximation of discrete distributions.

In reliability studies, it is important to determine the nature of the quantile function at specific quartiles especially in management or decision analysis. This is another area that most of the methods perform badly since the solutions are produced firstly before the quartiles are considered. This is cumbersome, especially when a specific result is desired in particular and not in the totality. Since QM depends heavily on ODE, it is also important to study the existence and uniqueness of the anticipated solution on a given domain. The approach could also be designed in a way to explain the instances where analytic solutions are possible in a distribution with intractable CDF.

The paper is organized as follows: section one is the introduction, section two is the details of the quantile function, their mathematical expression, examples and applications. Section three is the description and the importance of the inversion method. Section four gives the different methods of quantile approximation. Section five contains the historical development, applications, contributors, examples, advantages of the QM. Section six contains various critics and possible research areas of the QM approach. The paper ends with concluding remarks

2. Quantile function

The quantile function is the inverse cumulative distribution function (CDF). Cumulative distribution function of a random variable X, evaluated at a given point x is the probability that the random variable X will assume values less than or equal to the given point x.

Mathematically, the cumulative distribution function can be expressed as;

$$F(x) = P(X \leq x),\ x \in \mathrm{R} \tag{1}$$

Apart from the relationship between the CDF and quantile function, CDF can be used in the cumulative frequency analysis and statistical hypothesis testing.

For both discrete and continuous random distribution, the quantile function is generally known to be given as:

$$w(p) = F^{-1}(p) = \inf\{x \in \mathrm{R}: p \leq F(x)\} \tag{2}$$

with $p \in (0,1)$ and the cumulative distribution defined on the real line.

If the function F exists, continuous and is strictly monotone increasing, then the infinium function can be replaced by the minimum function, Eq. 1 becomes;

$$Q(p) = F^{-1}(p) \tag{3}$$

It must also be noted that the relationship between the CDF and quantile function is not like the one between a function and its respective inverse because the mapping of the CDF is not a one-to-one. This is because in general, the CDF is not necessarily monotone increasing, but only non-decreasing and as such, does not fully admits the inverse function. But most of the distributions are characterized with the CDF that is strictly monotone increasing.

The definitions are valid based on the assumption of the existence and uniqueness of the quartiles. This is due to that fact that the CDF that gave birth to them are right continuous and the measure of discontinuity is zero. Five-number summary (a, q_1, q_2, q_3, b) is obtained from the quantile function, where a and b are the minimum and maximum values of X. The five-number summary contains vital information about the center, spread, skewness and detection of outliers of the distribution of X.

The quantile functions of some distributions are enumerated.

1. Normal Distribution

$$Q(p) = \mu + \sigma\sqrt{2}erf^{-1}(2p-1) \tag{4}$$

2. Standard Normal Distribution

$$Q(p) = \sqrt{2}erf^{-1}(2p-1) \tag{5}$$

3. Exponential Distribution

$$Q(p) = -\frac{1}{\lambda}\ln(1-p) \tag{6}$$

4. Cauchy Distribution

$$Q(p) = \mu + \sigma tan\left[\pi(p-\tfrac{1}{2})\right]. \tag{7}$$

Another aspect of QF is the conditional QF defined by Parzen (2004). Given a random vector (X, Y);

$$Q_{Y|X} = \inf\{x:\ F_{Y|X} \geq p\}, 0 \leq p \leq 1 \tag{8}$$

where $F_{Y|X}$ is the conditional CDF for the random vector (X, Y).

Quantile function is the most studied probability function, largely because of the following:

i. It can be used to model and predict the percentiles, the quartiles especially median, which is more robust and resistant to outliers (Hampel, 1974).
ii. Measurements involving the QF are often less influenced by outliers or extreme observations.
iii. QF can be seen as an alternative to the CDF in analysis of lifetime probability models with heavy tails.
iv. Probability distributions whose statistical reliability measures do not have a close or explicit form can be conveniently represented through the QF.

Details of more on the theoretical and general applications of the QF can be found in Gilchrist (2007).

Generally, there seems to be three major areas where QF is mostly apply. These are; a) Value at risk (VAR): It is very useful in measuring risk in computational finance. For a given portfolio of a risky asset defined over a fixed time horizon t, let l denote the potential loss of the portfolio with a CDF given as; $F_l(x) = P(l \leq x)$, the VAR can be defined, for a given confidence interval, $0 < \alpha < 1$, as the smallest number x such that the probability of that the potential loss of the portfolio of a risky asset exceeds x is less than $1 - \alpha$ for the given fixed time horizon t. This can be represented mathematically as;

$$\text{VAR} = \inf\{x \in \mathrm{R} : P(l > x) \leq 1 - \alpha\} \quad (9)$$

$$\text{VAR} = \inf\{x \in \mathrm{R} : P(l \leq x) \geq \alpha\} = \inf\{x \in \mathrm{R} : F_l(x) \geq \alpha\} \quad (10)$$

It can be seen that the VAR is the α quantile of the loss CDF. VAR can be used with caution as its derelictions can manifest in several ways (McNeil et al., 2005).

b) Monte Carlo simulations. This arises from the fact that the quantile function of a given distribution maps the uniform variates to that given distribution and this has found substantial applications in computational finance and insurance (Luu, 2016).

The Quantile function of the following distribution can be simulated by replacing the p in Eqs. 4, 5, 6 and 7 with the standard continuous uniform distribution u = U (0, 1).

$$Q(u) = \mu + \sigma\sqrt{2}erf^{-1}(2u - 1) \quad (11)$$

$$Q(u) = \sqrt{2}erf^{-1}(2u - 1) \quad (12)$$

$$Q(u) = -\frac{1}{\lambda}\ln(1 - u) \quad (13)$$

$$Q(u) = \mu + \sigma tan\left[\pi(u - \tfrac{1}{2})\right]. \quad (14)$$

The inversion method is often used in this method.

This can be established with the following equations;

$$P(X \leq x) = P(Q(u) \leq x) = P\big(u \leq F(x)\big) = F(x). \quad (15)$$

c) Copulas: It is used to describe the dependence between random variables. It can be used to define the extent of which the marginal probability distributions may be linked, that is, the marginal probability distribution of each random variable is uniform. This forms the basis of the importance of quantile in copulas since the marginal distributions are uniform and can be linked to the quantile. This was a common assertion found in Franke et al. (2004).

Moreover, copulas are often used in quantitative finance such as price volatility analysis; price spread options analysis, value at risk forecasting, portfolio optimization, improving returns on investment estimation and statistical arbitrage (Low et al., 2013).

Other areas of applications are also available, such as: engineering and hydrology (Cai and Reeve, 2013), warranty data analysis (Wu, 2014), modeling turbulent combustion (Ruan et al., 2014), medicine (Eban et al., 2013), meteorology (Laux et al., 2011) and random vector generation (Bandara and Jayasumana, 2011).

Despite the usefulness of the QF, it cannot be used to model time dependent events unless it is either transformed or induced and also a measure of variability is restricted to inter-quartile range. Interpretation of multivariate quantile functions has been often cumbersome. Also simulation is often limited to uniform continuous distribution (Machado and Mata, 2005).

3. Inversion method

Evidence from scientific literature revealed that the inversion method is most sought after method of obtaining the QF (Korn et al., 2010). The inversion method is used mainly when the CDF of the distribution has closed form representations and the presence of shape parameters that defined such distribution does not have a computational effect on the outcome of the QF.

Some of the merits of the inverse methods are listed:

a) It is very simple method employed to generate random variables from given distributions.
b) Only one uniform distribution is required to generate non-uniform variates assuming that the CDF is continuous and monotone increasing.
c) Inversion method creates room for sampling from conditional probability distribution (Glasserman, 2013).
d) Inversion method is believed to be the only transformation technique method adaptable with various variance reduction methods such as sampling, random numbers generation and arithmetic random calculations (Gentle, 2003).

This further explains the role of the inversion method in inducing correlation among variates required in variance reduction techniques (Law and Kelton, 2000).

e) The inversion method enables the researcher to generate the maximum of a given sample and the order statistics in a proficient manner.
f) Researchers have unanimously concluded that the inversion method is the best available and scientifically proven method that preserves the structure of quasi random numbers better than other known methods like the Box Muller transform and Marsaglia transform (Galanti and Jung, 1997). However, Ökten and Göncü (2011) had rejected the assertion.

However, inversion method is handicapped when the CDF does not have a closed form representation. Some of the distributions whose CDF have no closed form includes: Normal, chi-squared, beta, gamma, student's t, F and so on. Secondly, the inversion method seems to be less computation feasible when compared with rejection sampling (Luu, 2016).

4. Quantile approximation

The unavailability of the closed form of the CDF of continuous probability distributions is the main rationale for several approximations available in scientific literature.

Generally, research on quantile approximation is divided into four categories, namely; series expansions, closed form or functional approximation, numerical algorithm and the closed form expression drafted in terms of the quantile function of another distribution.

It should be noted that aside from the aforementioned four categories, researchers have devised other inspiring and innovative means of approximating the quantile function based on the models they are working on. They can be regarded as hybrid methods that combine different methods of approximation with optimization methods. Some of which are listed: recovery of QF from the moments of the underlying distribution (Mnatsakanov and Sborshchikovi, 2017), approximation of higher quantile using the intermediate one (de Valk, 2016); artificial bee colony optimization and curve fitting methods (Kabalci, 2018). Others are estimates of the quantile from a given large samples of data that arrive in sequence, hence the quantile is approximated using a stochastic learning approach (Yazidi and Hammer, 2015) and approximation by kernel estimation method (Hasu et al., 2011).

4.1. Functional approximation

This is the use of a function that closely resembles the target function. Evidences from some selected work done in closed form approximation of quantile functions of probability distributions are focused on improving or error reduction of existing methods. Some selected works in the area are normal distribution (Soranzo and Epure, 2014) and Gompertz–Makeham distribution (Jodrá, 2009). The major drawback is the complexity of the functions and low computational speed.

4.2. Using the QF of one to approximate another

This is the use of the QF of a known distribution to derive the QF of another, if there is a tangible relationship between the two distributions (Munir, 2013).

4.3. Numerical techniques or algorithms

Numerical techniques and algorithms are often used in the absence of the closed form expression of the quantile function. Computational efficiency is the core advantage of this method (Lange, 2010). The numerical techniques or algorithms are further divided into the methods of root finding, rational approximation and interpolation.

In particular, some specialized numerical algorithms and techniques have been used for quantile approximations. Examples are: Hermite interpolation (Hörmann and Leydold, 2003) and regularization procedure (Chernozhukov et al., 2010).

4.3.1. Root finding

Standard numerical algorithms are used to obtain the zeros of the equation that linked the CDF to the QF. This can be achieved when the CDF is continuous and monotone increasing.

Newton and secant methods are mostly used here, but these methods suffer from slow convergence when applied to complex distributions (Hörmann et al., 2013). Bisection and Halley's methods are improvement over the Newton and secant methods (Huh, 1986). Their improvement is only on increasing the rate of convergence to the exact solutions of the required quantile functions.

Functional iteration or fixed point method is another root finding method which is an improvement over all other root finding methods. The merits of the fixed point method include: quadratic convergence, wide applicability, only the CDF and PDF are required, applicability to distributions on finite intervals and mixed distributions. Convergence problem subsist in this method, however a better result can be obtained through the use of Steffensen's acceleration technique (Minh and Farnum, 2010).

Root finding techniques require a formula of initial guess which is of high accuracy and tolerable accurate CDF algorithm. The method is very efficient (Dagpunar, 1989), however, their usefulness is often limited to the refining of the output of another algorithm and slow. Readers are referred to Acklam (2009). Moreover, they are applicable to distributions with shape parameters (Luu, 2016).

4.3.2. Rational approximation

The central theme of rational approximation is to approximate the QF by a function of the form;

$$Q(p) = \frac{c_0+c_1p+c_2p^2+\cdots+c_np^n}{d_0+d_1p+d_2p^2+\cdots+d_np^n} + e(p) \tag{16}$$

defined on an error e(p) satisfying a given bound.

There seems to be a general agreement to the difficulty of obtaining rational approximation using a single domain, so researchers resorted to splitting the domain into subdomains and subsequently obtaining several rational approximations (Acklam, 2009). The method has advantage of computational efficiency (Shaw et al., 2014); however it is plagued with low convergence and slow speed in computation.

4.3.3. Interpolation

This is curve fitting with the use of polynomials. This is often done by the use of numerical integration. A general procedure of approximating the quantile function of probability distributions was provided by Ahrens and Kohrt (1981) and Schiess and Matthews (1985). This involves three steps, namely: Numerical integration of the CDF in order to obtain the PDF, use the criteria of the behavior of the CDF to divide the domain of the quantile function into unequal length intervals and finally, determine the nature of the interpolating function at each subdomain or interval. Most of the results obtained from interpolation are better than both the rational and root finding methods. This was submission of Costa (2018). The method is not suitable for distributions with shape parameters. The process of initialization of each parameter at each computation slows the computation speed.

4.4. Series expansions

This is the use of a series or sequence of terms to approximate the quantile function. Most often asymptotic expansion is preferred. Asymptotic expansion is a formal series expansion of given function whose partial sums of the first few terms of the series can be a good approximation to the given function. Also asymptotic expansion can be extended to the use of a series expansion of a function to approximate a similar function based on the observed similarities of the functions. Several kinds of series expansions exist; they include: Taylor, Power, MacLaurin, Laurent, Dirichlet, Fourier series and so on. Series expansions methods of quantile approximations are listed: Cornish-Fisher expansions, orthogonal expansion and the quantile mechanics (QM).

4.4.1. Cornish fisher expansions

This is the most often used quantile approximation technique. Cornish-Fisher expansion is primarily used to obtain the approximation of the quantiles of a given probability distribution based on its cumulants. Cornish and Fisher (1938) and Fisher and Cornish (1960) defined for a given random variable X, with mean 0 and standard deviation 1, an expansion for the approximation of the pth quantile function $Q_X(p)$ of the standard normal distribution based on their cumulants;

$$\begin{aligned} Q_X(p) = Q_Z(p) + \frac{Q_Z(p)^2-1}{6}k_3 + \frac{Q_Z(p)^3-3Q_Z(p)}{24}k_4 \\ -\frac{2Q_Z(p)^3-5Q_Z(p)}{36}k_3^2 + \frac{Q_Z(p)^4-6Q_Z(p)^2+3}{120}k_5 \\ -\frac{Q_Z(p)^4-5Q_Z(p)^2+2}{24}k_3k_4 \\ +\frac{12Q_Z(p)^4-53Q_Z(p)^2+17}{324}k_3^3 \end{aligned} \tag{17}$$

where $Q_X(p)$ is the p-quantile of the standard normal distribution, $k_1, k_2, k_3, \ldots$ are the cumulants. It should be noted that the expansion can be extended to other probability distributions by normalization. Hill and Davis (1968) extended the works of Cornish-Fisher to arbitrary base distribution. This was further refined by Shaw et al. (2014).

Cornish Fisher is widely used in computational finance, for example, in the calculation of Value at Risk (VAR). However, it is limited in some ways. The quantile function is not inevitably monotone, approximations around the tail regions are cumbersome and the asymptotic nature connotes that increase of terms of the expansion does not necessarily improve the approximation. Moreover, Cornish-Fisher approximation performs better than the normal approximation (Jaschke, 2001).

Cornish Fisher expansions are ordinarily centered on the normal distribution, but regrettably, the normal distribution is not a universal model as it fails when applied to skewed data. This is worsened by the fact that most real life data are skewed. The method is intolerant to moderate or large departure from normality. The method is known to be not robust and have issues when the distribution is characterized by heavy tails and skewness. The consequences of the weaknesses are that the method may not always give reliable approximations (Coleman, 2012). The problems associated with Cornish Fisher expansions are unchanged despite some modifications of the method (Chernozhukov et al., 2010).

4.4.2. Orthogonal expansions

This method was proposed by Takemura (1983). The main idea was to find a base distribution from which it can be linked to an orthonormal basis of the second moment of the distribution. Orthogonal expansion makes use of the Fourier approach to express the target QF to that of the base QF similar to the Cornish Fisher expansion.

Convergence of series is an important advantage of orthogonal expansion over the Cornish Fisher expansion. However the method is limited to convergence to the second moment norm and difficulty of computing the terms of the series analytically.

4.4.3. Quantile mechanics

This is the use of differential equations to approximate the quantile function of a given probability distributions. Linear or nonlinear ordinary differential equations seem to be a common route been followed by researchers on the voyage of approximation of quantile functions (Hill and Davis, 1968). These happen to be the foundation behind the QM.

Researchers in this area maintained that the first order ODE of the quantile function is the reciprocal of the probability density function of the probability distribution (Soni et al., 2012). That is the PDF has to be expressed in terms of the quantile function with the assumptions that the PDF is differentiable or has a close form representation. This was an attempt to approximate the QF of many distributions having no close form expressions for their CDFs.

$$\frac{dw}{dp} = \frac{1}{f(x)} \qquad (18)$$

where w(p) is the quantile function expressed in terms of p, 0 < p < 1.

Different names have been given to the left hand side of Eq. 18, the quantile density function is the most often used (Soni et al., 2012) while the term "the Sparsity function" was used by Tukey (1965).

Consistent and efficient estimators for the left hand side of Eq. 18 have been derived from Babu and Rao (1990). In addition, other methods of estimating the quantile density function include: Kernel method (Falk, 1984), wavelets methods (Chesneau et al., 2016) and moments (Mnatsakanov and Sborshchikovi, 2017). The Quantile mechanics approach is a key method of using the ODE to approximate the QF.

5. Quantile mechanics

The major contribution of the review is to investigate the extent of which the QM approach has been applied in quantile approximation. The QM was introduced by Steinbrecher and Shaw (2008) being inspired by the earlier vague works done on the use of ODE in quantile approximations and a need to improve the use of quantile mechanics in computational finance.

5.1. Formulation

The objective of the QM approach was to modify Eq. 18 in such a way that a feasible solution can be obtained. That is a transmission from estimation to approximation. Given a CDF F and its related QF Q(p) for probability distributions whose CDF is monotone increasing and continuous; That is;

$$Q(p) = F^{-1}(p) \qquad (19)$$

where the function $F^{-1}(p)$ is the compositional inverse of the CDF.

Suppose the PDF f(x) is known and the differentiation exists. The first order quantile equation is obtained from the differentiation of Eq. 19 to obtain;

$$Q'(p) = \frac{1}{F'(F^{-1}(p))} = \frac{1}{f'(Q(p))} \qquad (20)$$

According to the submissions from literature, the probability density function is the derivative of the cumulative distribution function. The conditions being that the CDF is monotone increasing and the measure of discontinuities is approximately zero. The solution to Eq. 18 is often cumbersome as noted by Ulrich and Watson (1987). This is due to the nonlinearity of terms introduced by the density function f. Ulrich and Watson (1987) used Eq. 18 to obtain the quantile approximations of the normal, exponential, gamma and Cauchy distributions. In the same vein, Leobacher and Pillichshammer (2002) obtained the result for the hyperbolic distribution. This is given as;

$$Q'(p) = \frac{1}{F'(F^{-1}(p))} = \frac{1}{e} e^{a\sqrt{a^2+(F^{-1}(p)-c)^2 - b(F^{-1}(p)-c)}} \qquad (21)$$

The solution is cumbersome and expensive. In order to improve on the use of ODE in QF approximate, Steinbrecher and Shaw (2008) proposed a new approach termed the "Quantile mechanics".

Some algebraic operations are required to find the solution of Eq. 20. Moreover, Eq. 20 can be written as;

$$f(Q(p))Q'(p) = 1 \qquad (22)$$

Applying the product rule of differentiation to obtain;

$$\frac{d^2Q(p)}{dp^2} = V(Q(p))\left(\frac{dQ(p)}{dp}\right)^2 \qquad (23)$$

where the nonlinear term;

$$V(x) = -\frac{d}{dx}(\ln f(x)). \qquad (24)$$

These were the results of Steinbrecher and Shaw (2008).

5.2. Extensions and variants

Shaw and Brickman (2009) introduced the differential equation as modification or extension of Eq. 23;

$$f(w(p))\frac{d^2w(p)}{dp^2} + f'(w(p))\left(\frac{dw(p)}{dp}\right)^2 = 0 \qquad (25)$$

The differential equations obtained from quantile mechanics are cyclic in nature (Shaw et al., 2014).

Munir and Shaw (2012) noted that this approach is often difficult but has been unexplored, especially the solutions of Eq. 23. It is because many distributions have complicated probability density functions. They considered the following complex distributions: hyperbolic, α-stable, variance gamma and generalized inverse Gaussian distributions.

In order to solve the problem, they proposed a new method of transformation from the base distribution $Q_B(p)$ to the target distribution $Q_T(p)$. Assume a function C exists which associates or map the p^{th} quantile for the base distribution with the p^{th} quantile of the target distribution.

$$Q_T(p) = C\big(Q_B(p)\big) \quad (26)$$

An ODE describing the function C arises from the adoption of change of variable.

$$\frac{dQ_T(p)}{dp} = \frac{1}{f_T(Q_T(p))} \quad (27)$$

A change of variable is needed in order to apply the simple chain rule of differentiation. That is $x = Q_B(p)$. Eq. 27 is differentiated;

$$\frac{dQ_T(p)}{dp} = \frac{dC}{dx}\frac{dx}{dp} = \frac{dC}{dx}\frac{dQ_B}{dp} = \frac{dC}{dx}\frac{1}{f_B(Q_B(p))} = \frac{dC}{dx}\frac{1}{f_B(x)} \quad (28)$$

Substitute in Eq. 27 to obtain the first order recycling equation;

$$\frac{dC}{dx} = \frac{f_B(x)}{f_T(C(x))} \quad (29)$$

The equation can be solved thereafter for the chosen distribution. In the same line of research, Kleefeld and Brazauskas (2012) applied the quantile mechanics to derive the methods of trimmed moment estimators for the student's and gamma distributions. Inspired by the quantile mechanics approach, Cordeiro (2013) proposed a novel method of expressing the quantile function of any given beta generalized distribution.

Shaw and McCabe (2009) extended the quantile mechanics to the characteristic function of unknown probability distributions. The outcome was an integro-differential equation that assumes a power series solution despite the complexity of the nature of the unknown probability density function. This extension is motivated from the use of characteristic function of computing the CDF by inversion. This had help in applications in mathematical finance such as Shaw and Munir (2009) and Munir and Shaw (2012).

5.3. Applications

Quantile mechanics has been applied to some distributions. The details of the distributions where QM has been applied so far to the knowledge of the authors are summarized in Table 2.

Table 2: Authors contribution to the QM approach of quantile approximation of different distributions

Distribution	Authors
Beta	Steinbrecher and Shaw (2008)
Exponential	Shaw et al. (2014), Shaw and Brickman (2009)
Gamma	Steinbrecher and Shaw (2008), Luu (2016, 2015)
Generalized inverse Gaussian	Munir (2013), Munir and Shaw (2012)
Hyperbolic	Munir (2013), Shaw et al. (2014), Shaw and Brickman (2009), Munir and Shaw (2012)
Non-central chi square	Munir (2013)
Normal	Steinbrecher and Shaw (2008), Luu (2016), Shaw et al. (2014), Shaw and Brickman (2009), Shaw and McCabe (2009), Alu (2011)
Skew-normal	Luu (2016)
Snedecor's F	Munir (2013)
Student's t	Steinbrecher and Shaw (2008), Shaw et al. (2014), Shaw and Brickman (2009), Shaw and McCabe (2009)
Symmetric generalized Hyperbolic	Shaw and McCabe (2009)
Variance gamma	Munir (2013), Shaw et al. (2014), Shaw and Brickman (2009), Munir and Shaw (2012)
α stable	Munir (2013), Munir and Shaw (2012)
Some selected convoluted distributions	Okagbue et al. (2018)

5.4. Application to the normal distribution

The QF of the normal distribution is known as the probit function. The probit can easily be transformed from the CDF of the standard normal distribution using the inversion method, but the presence of the error function makes computation often expensive. The probit function is given by;

$$Q(p) = \sqrt{2}erf^{-1}(2p-1). \quad (30)$$

Steinbrecher and Shaw (2008) applied the QM approach to the normal distribution (standard) that is to obtain the approximation to Eq. 30. The details follow; the PDF of the standard normal distribution is given by;

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad (31)$$

transform to obtain;

$$f(w) = \frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}, \quad (32)$$

transforming using the direct approach of Eq. 18, to obtain;

$$\frac{dw}{dp} = \sqrt{2\pi}e^{\frac{w^2}{2}}, \quad (33)$$

differentiate again;

$$\frac{d^2w}{dp^2} = \sqrt{2\pi}e^{\frac{w^2}{2}}\frac{2w}{2}\frac{dw}{dp} = \sqrt{2\pi}e^{\frac{w^2}{2}}w\frac{dw}{dp}, \quad (34)$$

substitute Eq. 33 into Eq. 34;

$$\frac{d^2w}{dp^2} = w\left(\frac{dw}{dp}\right)^2, \quad (35)$$

with initial conditions:

$w(1/2) = 0, w'(1/2) = \sqrt{2\pi}$,

different approaches have been used to solve Eq. 35.

Steinbrecher and Shaw (2008) used the power series approach. The solution they obtained is given as:

$$w(p) = \sqrt{\frac{\pi}{2}}\sum_{k=0}^{\infty}\frac{d_k}{2k+1}(2p-1)^{2k+1} \quad (36)$$

where the coefficients d_k satisfies the stated nonlinear recurrence equation given by;

$$d_{k+1} = \frac{\pi}{4}\sum_{j=0}^{k}\frac{d_j d_{k-j}}{(j+1)(2j+1)} \quad (37)$$

with $d_0 = 1$, convergence is achieved for large k,

$\frac{d_{k+1}}{d_k} = 1$, as $k \to 0$.

The first few terms of series (Eq. 36) can be written as;

$$w(p) = -\frac{\sqrt{2\pi}}{2}\left(1+\frac{\pi}{12}\right) + \sqrt{2\pi}\left(1+\frac{\pi}{4}\right)p - \frac{\sqrt{2\pi}}{2}\pi p^2 + \frac{\sqrt{2\pi}}{3}\pi p^3 + \cdots \quad (38)$$

Alu (2011) used the modified Carleman embedding technique to obtain the solution of Eq. 36. The series solution was given as;

$$w(p) = -\frac{1}{2}\left(\frac{\sqrt{2\pi^3}}{12} + \sqrt{2\pi}\right) + \left(\frac{\sqrt{2\pi^3}}{4} + \sqrt{2\pi}\right)p - \frac{\sqrt{2\pi^3}}{2}p^2 + \frac{\sqrt{2\pi^3}}{3}p^3 + \cdots \quad (39)$$

Eqs. 38 and 39 are approximately the same and hence the approximate of the probit function. The equivalence of Eqs. 38 and 39 can be seen when their terms are compared.

5.5. Advantages of the quantile mechanics

The major contributions of the QM approach formed most of the merits over other forms of quantile approximations.

5.5.1. Superiority over other surveyed methods in approximating complex distributions

The QM is so far the best approach to the use of ODE in quantile approximation of probability distributions. Quantile approximations of complex distributions that can prove cumbersome to other methods have been obtained by the use of QM approach. The evidence can be seen in Munir and Shaw (2012) and Munir (2013) where they considered and consequently obtained the asymptotic representations of the following distributions: Generalized inverse Gaussian, Variance gamma, α-stable, hyperbolic and Snedecor's F distributions. However many distributions are remain unexplored.

5.5.2. Better approximations of distributions with shape parameters

As noted by Luu (2016) that quantile approximations of distributions with shape parameters are cumbersome and require two processes. The QM approach performed better than the Cornish Fisher expansion method in quantile approximation of the student's t distribution (Shaw and Brickman, 2009; Shaw and McCabe, 2009), gamma distribution (Steinbrecher and Shaw, 2008; Luu, 2015; 2016) and beta distribution (Steinbrecher and Shaw, 2008).

5.5.3. Precision and accuracy of results

The general quality of results, convergence to the exact values and high precision of the QM approach in quantile approximation presents the method as a major improvement over the others. This was the outcome in the approximation of the student's t distribution (Shaw and Brickman, 2009), normal distribution (Luu, 2016) and gamma distribution (Luu, 2015; 2016).

5.5.4. Speed and application to parallel computation

The QM approach has been used to develop numerical algorithms for quantile approximation, which was found to be efficient, robust, and fast and save computing time. For example the algorithm obtained for the normal QF is found to be better than those available in literature, see Shaw and Brickman (2009) and Luu (2015). The details on this aspect were discussed extensively in Luu (2016). In addition, the numerical algorithms are very suited for parallel computer architectures, precise and faster when compared with the existing ones. Subsequently, several research activities are expected to emerge here because of numerous distributions that are yet to be explored.

5.5.5. Application in momentum space

Quantile mechanics proved useful in quantile approximation of α-stable distributions. These are distribution for cases where only the characteristic function is known and the PDF and CDF is either unknown or does not have closed form representations (Munir, 2013; Shaw and McCabe, 2009). However the implementation is limited to high level languages and often slow in computation. This is because of the problem of reverting the power series in the computation.

5.5.6. PDF used in lieu of the CDF

QM approach makes use of the PDF which is more tractable for some distributions than the CDF. A survey of some distributions showed that the PDF is a lot more tractable than most CDF. Now, the intractability of the CDF is not an obstacle in the quantile approximation if the QM approach is applied.

6. Areas of possible studies

The QM approach can be seen as an important improvement over the other existing methods with some advantages evidenced by the research work done so far. However, the review has unearthed some striking issues bothering on the approach that will inevitably lead to further investigations in this research area.

6.1. Difficult approach for some distributions

The QM approach seems to be difficult in computation for some distributions. This is evidenced by the second order nonlinear ODE obtained for the Gompertz and Gumbel distributions. The solution of the two ODEs may prove a great challenge. In this case, the inversion method is a viable option. The reason why this kind of nonlinear ODE was obtained is subject for further studies.

6.2. Inconsistencies in solution of the concerned ODEs

Different series solution methods have been used in the solution of the nonlinear ODEs obtained through the QM approach. This includes: power series (Steinbrecher and Shaw, 2008) and Taylor series (Munir, 2013). The implications of the inconsistencies can manifest in two faced scenarios. Firstly, the comparison of the series solutions has not been done. This has prevented researchers from making general statements about their results. Secondly, this creates an avenue for the use of other methods available in scientific literature, thereby opening up the area for further research and criticism.

6.3. Complex for simple distributions

QM is a very complex approach of quantile approximation for simple distributions. Let us consider the exponential distribution. The quantile ODE whose solution is the QF of the exponential distribution is given by;

$$\frac{d^2w}{dp^2} = \lambda\left(\frac{dw}{dp}\right)^2. \quad (40)$$

Solution to Eqs. 40 is complex when compared with the use of inversion method that easily yields the QF of the exponential distribution given as;

$$Q_p = -\frac{\ln(1-p)}{\beta} \quad (41)$$

6.4. Derivatives of reciprocal of other probability functions

QM is limited to quantile function only. It remains to show what the nature of derivatives of the other probability functions would be. It is yet to be reported in scientific literature what the derivatives of the reciprocal of the following probability functions will be.

$$\frac{dy(x)}{dx} = \frac{1}{F(x)} \quad (42)$$

where F(x) is the CDF.

$$\frac{dy(x)}{dx} = \frac{1}{S(x)} \quad (43)$$

where S(x) is the Survival function.

$$\frac{dV(p)}{dp} = \frac{1}{V(p)} \quad (44)$$

where V(p) is the Inverse survival function.

$$\frac{dy(x)}{dx} = \frac{1}{h(x)} \quad (45)$$

where h(x) is the Hazard function

$$\frac{dy(x)}{dx} = \frac{1}{j(x)} \quad (46)$$

where j(x) is the Reversed hazard function

$$\frac{dy(x)}{dx} = \frac{1}{o(x)} \quad (47)$$

where o(x) is the Odd function.

6.5. Comparison between first and second order ODEs of the Quantile mechanics

The detailed comparison between Eqs. 20 and 23 has not been considered. The comparison is important to determine the extent of which Eq. 23 is an improvement over Eq. 20.

6.6. Comparison with the ODE obtained from ordinary derivative

Ordinarily the derivative of the PDF can be either used to obtain an ODE whose solution is the PDF of the distribution or the mode of the distribution. Likewise, the same can be applied to the QF. The ODE obtained through this process is different from the one obtained from QM. The reason of the discrepancies and detailed comparison is subject to further investigations by researchers.

An example of the ODE derived for the QF of the exponential distribution.

The Quantile function (QF) of the exponential distribution is given by;

$$Q(p) = -\frac{\ln(1-p)}{\lambda}. \quad (48)$$

The first order differential equation for the QF can be obtained from the differentiation of Eq. 48;

$$Q'(p) = \frac{1}{\lambda(1-p)}. \quad (49)$$

The condition necessary for the existence of the equation is $\lambda > 0, 0 < p < 1$. Simplify Eq. 49 to obtain;

$$\lambda(1-p)Q'(p) = 1 \quad (50)$$

The first order ordinary differential for the Quantile function of the exponential distribution is given as;

$$\lambda(1-p)Q'(p) - 1 = 0 \quad (51)$$
$$Q(0.1) = \frac{0.1054}{\lambda}. \quad (52)$$

To obtain the second order differential equation, differentiate Eq. 49 to obtain;

$$Q''(p) = \frac{1}{\lambda(1-p)^2}. \quad (53)$$

The condition necessary for the existence of the equation is $\lambda > 0, 0 < p < 1$. Simplify Eq. 53 to obtain;

$$(1-p)\left[\frac{1}{\lambda(1-p)^2}\right] = \frac{1}{\lambda(1-p)} \quad (54)$$
$$(1-p)Q''(p) = Q'(p) \quad (55)$$

The second order ordinary differential for the Quantile function of the exponential distribution is given as;

$$(1-p)Q''(p) - Q'(p) = 0 \quad (56)$$
$$Q'(0.1) = \frac{10}{9\lambda}. \quad (57)$$

Are the solutions the same and feasible? Will this applies to all distributions whose QFs are differentiable? Do the ODEs obtained through ordinary derivative give the same solution? Does it save time to work with the ones that gave linear ODEs against the QM approach that yielded nonlinear ODE? Why are the initial value conditions different? If the ordinary derivatives are available, do we still need QM approach and does QM give better result? These and more are expected to be subjects for further investigation.

6.7. Non-uniformity of initial or boundary conditions

Different initial or boundary conditions have been used interchangeably in the solution of the nonlinear ODEs obtained for the QF through the QM. Initial, center and boundaries have been applied (Steinbrecher and Shaw, 2008). One common feature used by the different authors is the use of change of variable to convert the boundary/initial conditions to a simpler one.

6.8. Difficulty of transformation of extended distributions

The transformation from base distributions to the target one may be cumbersome for extended probability distributions like truncated, extended, generalized, exponentiated, transmuted, weighted, inflated, inverted or compounded distributions. The details are subject of further research.

6.9. Extension to convoluted, ratio and product distributions

The QM has not been applied to the quantile approximation of convoluted, ratio and product probability distributions. These classes of distributions are used to model real life phenomena. An example is given to drive home the point.

This will attract research interest because of the diverse applications of convoluted, ratio and product distributions of probability distributions whose CDF does not have close form representation.

6.10. Non applicability to discrete probability distributions

The QM has joined the queue of the methods that cannot be routinely applied to the quantile approximation of discrete probability distributions. The search for such method goes on.

6.11. Multivariate and mixed Quantile approximations

This was briefly discussed by Steinbrecher and Shaw (2008). This remains unexplored largely because of difficulty in formulating efficient initial or boundary conditions required in the determination of the solution of the ODE obtained. The application of the QM approach for quantile approximation of mixed distributions has not been reported in literature. This area remains largely unexplored.

6.12. Alternatives to the uniform distribution

One of the major reasons for the quantile approximation is because of the use of QF in random number generation. This is because the QF maps the uniform variates to the target or desired distribution. Now that the QM approach have proved to be very efficient. Other alternatives to the uniform variates need to be considered. The recommended variates are somewhat related to the uniform variates.

The distributions are: u-quadratic, triangular, rectangular and trapezoidal distributions. The same need to be considered in copulas.

6.13. Region of analytic solutions

The QM is silent about the behavior of the method and the QF where the CDF of distributions considered has close functions. For example, the CDF of the Chi-square distribution has a close form representation at degrees of freedom equals two and the CDF of Erlang distribution has closed form at degrees of freedom equals to one.

6.14. Quantile function considered in totality

The QM has performed effectively in the quantile approximation for the tail probability of different probability distributions. The performance of the QM on different quartiles or percentage points has not been fully considered. The quantile is considered in totality (in most cases) without consideration of the lower and upper quartiles and so on.

6.15. Existence and Uniqueness of solutions

The outcome of the QM is a second order nonlinear ODE. The existence and the conditions that determine and guarantee the unique solution in the given domain have not been considered. This is expected to attract research interest, especially from mathematicians.

7. Conclusion

The review has attempted to present different views as regards to the quantile approximations in general and quantile mechanics in particular. The Quantile mechanics is a very efficient approach of using the ordinary differential equations to approximate the quantile functions of distribution with no close form for their cumulative distribution. Different areas of strength, weaknesses, limitations and unexplored areas of the Quantile mechanics approach were discussed based on the evidence obtained from the scientific literature available from that area of research. Quantile mechanics and other methods surveyed in this paper will continue to be relevant because of the wide applicability of the quantile function as seem in the review. The review has merged different views on quantile approximations which ordinarily approximates to the truth that this is the direction that research in that area is likely to go. Finally, some of the gaps observed in this paper have motivated to the proposal of different ordinary differential equations whose solutions are the probability functions of the studied probability distributions (Okagbue et al., 2017a, b, c).

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Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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