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On the extension of generalized Fibonacci function

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ABSTRACT

The Fibonacci sequence is well known for having many hidden patterns within it. The famous mathematical sequence 1,1,2,3,5,8,13,21,34,55,89,. . m, n, m +n . . . known as the Fibonacci sequence, $F_{n+1} = F_n + F_{n-1}, n \geq 1, F_1 = F_2 = 1$. It has been discovered in many places such as nature, art and even in music. It has an incredible relationship with the golden ratio. In this paper, we define Fibonacci function on real number field for all real x,f: R → R, there exist $f(x+n) = a\, f(x+n-1) + b\, f(x+n-2)$. We developed the notion of generalized Fibonacci function using the concept of Binet's formula and induction technique and construct the relation between generalized Fibonacci function and generalized Fibonacci numbers. We also develop the notion of generalized Fibonacci functions with period s using the concept of f -even and f -odd functions.

1. Introduction

The Fibonacci sequence is named after Leonardo of Pisa, who was known as Fibonacci. Fibonacci's 1202 book Liber Abaci introduced the sequence to Western European mathematics, although the sequence had been described earlier in Indian mathematics. A problem in third section of Liber Abaci led to the introduction of the Fibonacci sequence. The first fourteen terms are the numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377. Fibonacci sequence of numbers F_n defined by

$$F_{n+1} = F_n + F_{n-1}, n \geq 1, F_1 = F_2 = 1 \quad (1)$$

This sequence in which each number is the sum of two preceding numbers has proved extremely fruitful and appears in many different areas of mathematics and science. Fibonacci sequence is a popular topic for mathematical enrichment and popularization. It is the formula for a host of interesting and surprising properties. Fibonacci numbers have been studied in many different forms for centuries and the literature on the subject is vast.

One of the remarkable qualities of these numbers is the diversity of mathematical models where they play some sort of role and where their properties are of importance in explaining the ability of the model under discussion to explain whatever implications are characteristic in it. The fact that the ratio of successive Fibonacci numbers approaches the Golden ratio. The Fibonacci sequence has been generalized in a number of ways.

Fibonacci Function: A Function f defined on the real numbers is said to be Fibonacci Function, if it satisfies the following

$$f(x+2) = f(x+1) + f(x), \forall x \in R \quad (2)$$

where R is the set of real numbers.

2. Preliminaries

Firstly Elmore (1967) found the important result of Fibonacci function. Parker (1968) discussed the derivation of the Fibonacci functions exist and are easily find. It leads to more relation involving Fibonacci numbers.

In the research article Gandhi (2012) a usual extension of the Fibonacci sequence was proposed. Fibonacci function on real number field was defined

$$\forall\, x \in R, f: R \to R, \exists$$
$$f(x+n) = F_n\, f(x+1) + F_{n-1}\, f(x) \quad (3)$$

and also defined the limit value of Fibonacci function which is closed to 1.618. Now we define

Definition 2.1: Generalized Fibonacci sequence

$$F_n = a\, F_{n-1} + bF_{n-2}, n \geq 2, F_0 = F_1 = 1 \quad (4)$$

where a, b and n are integers.

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Definition 2.2: Generalized Fibonacci function**.** If f is differentiable function and f satisfies

$$f(x+n) = a\,f(x+n-1) + b\,f(x+n-2) \quad (5)$$

Then f is called generalized Fibonacci function.

Theorem 2.3: If $f(x+n)$ is a generalized Fibonacci Function, then $f'(x+n)$ and $\int f(x+n)\,dx$ are also.

Proof: we have

$$f(x+n) = a\,f(x+n-1) + bf(x+n-2)$$

then

$$f'(x+n) = a\,f'(x+n-1) + b\,f'(x+n-2)$$

and

$$\int f(x+n)\,dx = \int[a\,f(x+n-1) + bf(x+n-2)]dx$$
$$= a\int f(x+n-1)dx + b\int f(x+n-2)dx$$

Corollary2.4: If $f(x+n)$ and $g(x+n)$ are generalized Fibonacci Function, Then their sum is also.

Corollary 2.5: If $f(x+n)$ is generalized Fibonacci Function, Then $c\,f(x+n)$ is also generalized Fibonacci Function, where c is real constant.

Example2.6: Let $f(x) = m^x$ be a generalized Fibonacci function on R, when $m > 0$, then $m^{x+n} = a\,m^{x+n-1} + b\,m^{x+n-2}$.

We have, generalized Fibonacci function

$$f(x+n) = a\,f(x+n-1) + bf(x+n-2)$$

and $f(x) = m^x$ be a generalized Fibonacci function on R, then

$$m^2 = a\,m + b$$
$$\Rightarrow m^2 - a\,m - b = 0$$
$$\Rightarrow m = \frac{a \pm \sqrt{a^2+4b}}{2}$$

Hence $f(x) = \left[\frac{a\pm\sqrt{a^2+4b}}{2}\right]^x$ is a generalized Fibonacci function.

Example 2.7: Let f be a Generalized Fibonacci Function, if we define $g(x) = f(x+t)$, where $t \in R, \forall\, x \in R$ then g is also a generalized Fibonacci function.

Proof: It is given that $\forall\, x \in R$

$$g(x+n) = f(x+n+t)$$
$$\Rightarrow a\,f(x+t+n-1) + b\,f(x+t+n-2)$$
$$\Rightarrow a\,g(x+n-1) + b\,g(x+n-2)$$

Theorem 2.8: Let $f(x)$ be generalized Fibonacci function and let F_n be a sequence of Fibonacci number with $F_0 = 0, F_1 = 1$ then $,\forall x \in R$ & $n \geq 2$, an integer

$$f(x+n) = a\,F_n\,f(x+1) + b\,F_{n-1}\,f(x) \quad (6)$$

Proof: we have

$$f(x+n) = a\,f(x+n-1) + bf(x+n-2)$$

I*f n=2, then*
$$f(x+2) = a\ f(x+1) + b\,f(x)$$
$$\Rightarrow a\,F_2\,f(x+1) + b\,F_1\,f(x)$$
$$\Rightarrow true\ for\ n = 2$$
If n = 3, then

$$f(x+3) = a\,f(x+2) + b\,f(x+1)$$
$$\Rightarrow a\,\{a\,F_2\,f(x+1) + bF_1\,f(x)\} + b\,\{a\,F_1\,f(x+1) + bF_0\,f(x)\}$$
$$\Rightarrow a\,\{a\,F_2\,f(x+1) + bF_1\,f(x)\} + b\,\{a\,F_1\,f(x+1) + bF_0\,f(x)\}$$
$$\Rightarrow a\,\{a\,F_2 + bF_1\,\}\,f(x+1) + b\,\{a\,F_1 + bF_0\}\,f(x)$$
$$\Rightarrow a\,F_3\,f(x+1) + b\,F_2\,f(x)$$

If n = k + 1 then
$$f(x+k+1) = a\,f(x+k) + bf(x+k-1)$$
$$\Rightarrow a\,\{a\,F_k\,f(x+1) + bF_{k-1}\,f(x)\} + b\,\{a\,F_{k-1}\,f(x+1) + bF_{k-2}\,f(x)\}$$
$$\Rightarrow a\,F_{k+1}\,f(x+1) + b\,F_k\,f(x)$$

This proof is completed.

Theorem 2.9: If $f(x+n)$ is Generalized Fibonacci function then $\lim_{x\to\infty}\frac{f(x+n)}{f(x+n-1)} = \frac{a+\sqrt{a^2+4b}}{2}$

Proof: Let

$$m = \lim_{x\to\infty}\frac{f(x+n)}{f(x+n-1)} \Rightarrow \lim_{x\to\infty}\frac{f(x+n)}{f(x+n-1)} = \lim_{x\to\infty}\frac{a\,f(x+n-1)+bf(x+n-2)}{f(x+n-1)}$$
$$= a + b\lim_{x\to\infty}\frac{f(x+n-2)}{f(x+n-1)}$$
$$= a + b\lim_{x\to\infty}\frac{f(x+n-1)}{f(x+n)},$$

because of the property of golden ratio of Fibonacci sequence

$$= a + b\lim_{x\to\infty}\left[\frac{f(x+n)}{f(x+n-1)}\right]^{-1} \Rightarrow m = a + bm^{-1},$$

which gives

$$m = \frac{a+\sqrt{a^2+4b}}{2}$$

Theorem 2.10: If F_n is the sequence of generalized Fibonacci numbers with $F_0 = 0, F_1 = 1$, then

$$\left[\frac{a\pm\sqrt{a^2+4b}}{2}\right]^n = aF_n\left[\frac{a\pm\sqrt{a^2+4b}}{2}\right] + bF_{n-1} \quad (7)$$

Proof: from example 1, we have noticed

$$f(x) = m^x = \left[\frac{a\pm\sqrt{a^2+4b}}{2}\right]^x$$

is generalized Fibonacci function. Let

$m = \left[\frac{a \pm \sqrt{a^2+4b}}{2}\right]$

by applying theorem

$f(x+n) = a\,F_n\,f(x+1) + b\,F_{n-1}\;f(x)$

we have

$$m^{x+n} = f(x+n) = a\,F_n\,f(x+1) + b\,F_{n-1}\;f(x)$$
$$\Rightarrow m^{x+n} = a\,F_n\,m^{x+1} + b\,F_{n-1}\;m^x$$
$$\Rightarrow m^{x+n} = a\,F_n\,m^{x+1} + b\,F_{n-1}\;m^x$$
$$\Rightarrow m^n = a\,F_n\,m + b\,F_{n-1}$$
$$\Rightarrow \left[\frac{a \pm \sqrt{a^2+4b}}{2}\right]^n = aF_n\left[\frac{a \pm \sqrt{a^2+4b}}{2}\right] + bF_{n-1}$$

This proof is completed.

3. Generalized Fibonacci function in exponential form for complex variable

Now we look for generalized Fibonacci functions of the form $y = e^{zx}$, where z is complex. Then

$$f(x+n) = a\,f(x+n-1) + bf(x+n-2)$$
$$\Rightarrow e^{z(x+n)} = ae^{z(x+n-1)} + be^{z(x+n-2)}$$
$$\Rightarrow e^{z(x+n)}[1 - ae^{-z} - be^{-2z}] = 0$$
$$\Rightarrow e^{2z} - ae^z - b = 0 \tag{8}$$
$$\Rightarrow e^z = \frac{a \pm \sqrt{a^2+4b}}{2} = m$$
$$Let\; e^{z_1} = \frac{a+\sqrt{a^2+4b}}{2} = \gamma \; and\; e^{z_2} = \frac{a-\sqrt{a^2+4b}}{2} = \delta$$
$$Let\; z_1 = a_1 + ib_1 \; and\; e^{z_1} = e^{a_1+ib_1} = e^{a_1}e^{ib_1}$$
$$\Rightarrow e^{a_1}[cosb_1 + i\,sinb_1] = \gamma,\, since\; \gamma > 0.$$
$$So\; a_1 = log\,\gamma \; and\; b_1 = 2k\pi$$
$$Let\; z_2 = a_2 + ib_2 and\; e^{z_2} = e^{a_2+ib_2} = e^{a_2}e^{ib_2}$$
$$\Rightarrow e^{a_2}[cosb_2 + isinb_2] = \;\delta\; Since\; \delta < 0.$$
$$So\; a_2 = log\,\delta \; and\; b_2 = (2k+1)\pi$$

it can be easily seen that

$$\gamma\delta = b = e^{z_1}\cos 2k\, e^{z_2}\cos(2k+1) = |e^{z_1}||e^{z_2}|$$
$$\Rightarrow z_2 = -log\gamma \; or\; z_2 = -z_1$$

and two solutions of

$f(x+n) = a\,f(x+n-1) + bf(x+n-2)$

are

$y(x) = e^{zx}\cos 2kx$

and

$y(x) = e^{-zx}\cos(2k+1)x$

applying the linear condition, then we have

$$y(x) = c_1e^{zx}\cos 2kx + c_2\,e^{-zx}\cos(2k+1)x, \tag{9}$$

where $z = log\gamma$, k is any integer

$$\Rightarrow y(x) = c_1e^{(z+2k\pi i)x} + c_2\,e^{(-z+(2k+1)\pi i)x} \tag{10}$$

now from (8) we can find some interesting and useful results.

$$e^{2(z+2k\pi i)} - ae^{(z+2k\pi i)} - b = 0$$
$$\Rightarrow e^{2z} = ae^z + b$$

and similarly

$$e^{-2z} = b - ae^{-z}$$

also we can have beautiful relation

$$e^z + e^{-z} = a = \gamma + \delta.$$

We know first two terms of Fibonacci sequence are 0 and 1 .so after applying the conditions $y(0) = 0, y(1) = 1$ in (9), we have

$$c_1 = \frac{1}{\sqrt{a^2+4b}} \; and\; c_2 = -\frac{1}{\sqrt{a^2+4b}}$$

hence

$$y(x) = \frac{e^{(z+2k\pi i)x} - e^{(-z+(2k+1)\pi i)x}}{\sqrt{a^2+4b}}$$

Remark: If k = 0 and a = b =1 all results come true, which was given by Spickerman (1970).

Recently, many researchers Sroysang (2013) and Han et al. (2012) have dedicated their research to the study of several properties of the Fibonacci function. They presented some properties on the Fibonacci functions with period k using the concept of f-even and f-odd functions with period k. Moreover, they also established some properties on the odd Fibonacci functions with period k.

Here first we define generalized Fibonacci function with period s and based examples.

4. Generalized Fibonacci function with period s

Let s be a positive integer, A function $g: R \to R$ is said to generalized Fibonacci function with period s if

$$g(x+ns) = ag\{x + (n-1)s\} + b\,g\{x + (n-2)s\}, \forall x \in R$$

Example4.1: Let $g(x) = m^{\frac{x}{s}}$ be a generalized Fibonacci function with period $s \in N, m > 0$, then

$$m^{\frac{x+ns}{s}} = am^{\frac{x+(n-1)s}{s}} + bm^{\frac{x+(n-2)s}{s}}$$
$$\Rightarrow m^2 - am - b = 0$$
$$\Rightarrow m = \left[\frac{a+\sqrt{a^2+4b}}{2}\right]$$

then

$$g(x) = \left[\frac{a+\sqrt{a^2+4b}}{2}\right]^{\frac{x}{s}}$$

Corollary 4.2: Let $g: R \to R$ be a generalized Fibonacci function with period $s \in N$, Assume that g is differentiable then g' is also generalized Fibonacci function with period s.

Corollary 4.3: Let $g: R \to R$ be a generalized Fibonacci function with period $s \in N$ and let $f(x) =$

$g(x+t), \forall x \in R, t \in R$, then f is also generalized Fibonacci function.

$$f(x+ns) = g(x+ns+t)$$
$$g(x+ns) = ag\{x+(n-1)s+t\} + bg\{x+(n-2)s+t\}$$
$$= a\,f\{x+(n-1)s\} + b\,f\{x+(n-2)s\}$$

Example 4.4: Let $s \in N$ and $t \in R$ and let $f: R \to R$, defined by $f(x) = \left[\frac{a+\sqrt{a^2+4b}}{2}\right]^{\frac{x+t}{s}}, \forall x \in R$, then f is generalized Fibonacci function with period s.

Theorem 4.5: Let $f: R \to R$ be a generalized Fibonacci function with period $s \in N$ and let $F_n = a\,F_{n-1} + bF_{n-2}$, $n \geq 2$ with $F_0 = F_1 = 1$, then

$$f(x+ns) = a\,F_{n-1}\,f(x+s) + b\,F_{n-2}\,f(x) \quad (11)$$

Proof: theorem can be proved easily by induction method. Induction method plays a vital role in Fibonacci sequence.

For n = 2

$$f(x+2s) = a\,F_1\,f(x+s) + b\,F_2\,f(x)$$

is obviously true. Now assume that result is true for n = n

$$f(x+ns) = a\,F_{n-1}\,f(x+s) + b\,F_{n-2}\,f(x).$$

Then for n = n +1

$$f(x+(n+1)s) = a\,f\{x+ns\} + b\,f\{x+n-1)s\}$$
$$\Rightarrow f(x+(n+1)s) = a[a\,F_{n-1}\,f(x+s) + b\,F_{n-2}\,f(x)] + b[a\,F_{n-2}\,f(x+s) + b\,F_{n-3}\,f(x)$$
$$= a[a\,F_{n-1} + bF_{n-2}]f(x+s) + b[a\,F_{n-2} + bF_{n-3}]f(x)$$
$$= a\,F_n\,f(x+s) + b\,F_{n-1}\,f(x)$$

hence the theorem. Now we define more terms for further results.

5. Generalized odd Fibonacci function with period s

Let s be a positive integer. A function $g: R \to R$ is called generalized odd Fibonacci function with period s, if it satisfies following condition.

$$g(x+ns) = -a\,g\{x+(n-1)s\} + b\,g\{x+(n-2)s\}, \forall x \in R$$

Corollary 5.1: if $g: R \to R$ be generalized odd Fibonacci function with period s.then g' is also generalized odd Fibonacci function.

Corollary 5.2: Let $g: R \to R$ be generalized odd Fibonacci function with period s and $f(x) = g(x+t), \forall x \in R, t \in R$,then f(x) is also generalized Fibonacci Function with period s.

Sroysang (2013) has given two definitions of f – even and f- odd functions with period s. here we restate them to give more advanced results.

Definition 5.3: f-even functions with period s. Let s be appositive integer and $\alpha: R \to R$ be such that if

$$\alpha\{x+ns\} = \alpha\{x+(n-1)s\} = \alpha\{x+(n-2)s\} = \cdots = \alpha(x),$$

then α is said to be f even function with period s.

Definition 5.4: f-odd functions with period s. Let s be a positive integer and $\alpha: R \to R$ be such that if

$$\alpha\{x+ns\} = -\alpha\{x+(n-1)s\} = -\alpha\{x+(n-2)s\} = \cdots = -\alpha(x),$$

then α is said to be f - odd function with period s.

Example 5.5: Let $g(x) = m^{\frac{\mu x}{s}}$ be a generalized Fibonacci function with period $s \in N, m > 0$, then

$$m^{\frac{\mu(x+ns)}{s}} = -am^{\frac{\mu x(+(n-1)s)}{s}} + bm^{\frac{\mu(x+(n-2)s)}{s}}$$
$$\Rightarrow m^2 + am - b = 0$$
$$\Rightarrow m = \left[\frac{-a+\sqrt{a^2+4b}}{2}\right]$$

then

$$g(x) = \left[\frac{-a+\sqrt{a^2+4b}}{2}\right]^{\frac{x}{s}}$$

Theorem 5.6: Let $s \in N, \alpha: R \to R$ be f-even function with period s and $\varphi: R \to R$ be continuous function, then φ is generalized Fibonacci function with period s, if and only if αφ is generalized Fibonacci function with period s.

Proof: Assume that for any $x \in R$, φ is generalized Fibonacci function with period s, then

$$(\alpha\varphi)(x+ns) = \alpha(x+ns)\varphi(x+ns)$$
$$= \alpha(x+(n-1)s)\,\varphi(x+ns)$$
$$\Rightarrow \alpha(x+(n-1)s)[a\,\varphi\{x+(n-1)s\} + b\varphi\,\{x+(n-2)s\}]$$
$$\Rightarrow a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-1)s)\,\varphi\,\{x+(n-2)s\}]$$
$$\Rightarrow a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-1)s)\,\varphi\,\{x+(n-2)s\}]$$

α is f-even function, so it implies that

$$\Rightarrow a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-2)s)\,\varphi\,\{x+(n-2)s\}]$$
$$\Rightarrow a\,(\alpha\varphi)\{x+(n-1)s\} + b\,(\alpha\varphi)\,\{x+(n-2)s\} \Rightarrow \alpha\varphi$$

is a generalized Fibonacci function with period s. Now, assume that αφ is a generalized Fibonacci function with period s and let $x \in R$, then

$$\alpha\varphi(x+ns) = a\,(\alpha\varphi)\{x+(n-1)s\} + b\,(\alpha\varphi)\,\{x+(n-2)s\}$$
$$\Rightarrow a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-2)s)\,\varphi\,\{x+(n-2)s\}]$$

α is f-even function so it implies that

$$\Rightarrow a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-1)s)\,\varphi\,\{x+(n-2)s\}]$$
$$= \alpha(x+(n-1)s)[a\,\varphi\{x+(n-1)s\} + b\varphi\{x+(n-2)s\}]$$
$$\Rightarrow \alpha(x+ns)[a\,\varphi\{x+(n-1)s\} + b\,\varphi\{x+(n-2)s\}]$$

$\Rightarrow \varphi$ is generalized Fibonacci function.

the proof is completed. Similarly another theorem can also be proved.

Theorem 5.7: Let $s \in N, \alpha: R \to R$ be f-even function with period s and $\varphi: R \to R$ be continuous function, then φ is generalized odd Fibonacci function with period s, if and only if $\alpha\varphi$ is generalized odd Fibonacci function with period s.

Proof: First, we suppose that, φ is generalized odd Fibonacci function with period s, then for any $x \in R$, we have

$(\alpha\varphi)(x+ns) = \alpha(x+ns)\varphi(x+ns)$
$= \alpha(x+(n-1)s)\,\varphi(x+ns)$
$\Rightarrow \alpha(x+(n-1)s)[-a\,\varphi\{x+(n-1)s\} + b\varphi\,\{x+(n-2)s\}]$
$\Rightarrow -a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-1)s)\,\varphi\,\{x+(n-2)s\}]$
$\Rightarrow -a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-1)s)\,\varphi\,\{x+(n-2)s\}]$

α is f-even function, so it implies that

$\Rightarrow -a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-2)s)\,\varphi\,\{x+(n-2)s\}]$
$\Rightarrow -a\,(\alpha\varphi)\{x+(n-1)s\} + b\,(\alpha\varphi)\,\{x+(n-2)s\}$
$\Rightarrow\ \alpha\varphi$

is a generalized odd Fibonacci function with period s.

Now, we assume that $\alpha\varphi$ is a generalized odd Fibonacci function with period s and let $x \in R$, then

$\alpha\varphi(x+ns) = -a\,(\alpha\varphi)\{x+(n-1)s\} + b\,(\alpha\varphi)\,\{x+(n-2)s\}$
$\Rightarrow -a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-2)s)\,\varphi\,\{x+(n-2)s\}]$

α is f-even function so it implies that

$\Rightarrow -a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-1)s)\,\varphi\,\{x+(n-2)s\}]$
$= \alpha(x+(n-1)s)[-a\,\varphi\{x+(n-1)s\} + b\varphi\{x+(n-2)s\}]$
$\Rightarrow \alpha(x+ns)[-a\,\varphi\{x+(n-1)s\} + b\,\varphi\{x+(n-2)s\}]$
$\Rightarrow \varphi$ is generalized odd Fibonacci function with period s.

this proof is completed.

Theorem 5.8: Let $s \in N, \alpha: R \to R$ be f-odd function with period s and $\varphi: R \to R$ be continuous function, then φ is generalized Fibonacci function with period s, if and only if $\alpha\varphi$ is generalized odd Fibonacci function with period s.

Theorem 5.9: Let $s \in N, \alpha: R \to R$ be f-odd function with period s and $\varphi: R \to R$ be continuous function, then φ is generalized odd Fibonacci function with period s, if and only if $\alpha\varphi$ is generalized Fibonacci function with period s.

Proof: First, we suppose that, φ is generalized odd Fibonacci function with period s, then for any $x \in R$, we have

$(\alpha\varphi)(x+ns) = \alpha(x+ns)\varphi(x+ns)$
$= -\alpha(x+(n-1)s)\,\varphi(x+ns)$
$\Rightarrow -\alpha(x+(n-1)s)[-a\,\varphi\{x+(n-1)s\} + b\varphi\,\{x+(n-2)s\}]$
$\Rightarrow a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} - b\,\alpha(x+(n-1)s)\,\varphi\,\{x+(n-2)s\}]$

α is f-even function, so it implies that

$\Rightarrow a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-2)s)\,\varphi\,\{x+(n-2)s\}]$
$\Rightarrow a\,(\alpha\varphi)\{x+(n-1)s\} + b\,(\alpha\varphi)\,\{x+(n-2)s\}$
$\Rightarrow\ \alpha\varphi$ is a generalized Fibonacci function with period s.

Now, we assume that $\alpha\varphi$ is a generalized odd Fibonacci function with period s and let $x \in R$, then

$\alpha(x+(n-1)s)\varphi(x+ns) = -\alpha(x+ns)\varphi(x+ns)$
$\Rightarrow -\alpha\varphi(x+ns) = -[a\,(\alpha\varphi)\{x+(n-1)s\} + b\,(\alpha\varphi)\,\{x+(n-2)s\}]$
$\Rightarrow -a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} - b\,\alpha(x+(n-2)s)\,\varphi\,\{x+(n-2)s\}]$

α is f-odd function so it implies that

$\Rightarrow -a\,\alpha(x+(n-1)s)\,\varphi\{x+(n-1)s\} + b\,\alpha(x+(n-1)s)\,\varphi\,\{x+(n-2)s\}]$
$= \alpha(x+(n-1)s)[-a\,\varphi\{x+(n-1)s\} + b\varphi\{x+(n-2)s\}]$
$\Rightarrow \varphi$ is generalized odd Fibonacci function with period s.

the proof is completed.

Example 5.10. Let s be a positive odd integer. We define

$$\alpha = \sin \pi x \text{ and } g\,(x) = \left[\frac{a+\sqrt{a^2+4b}}{2}\right]^{\frac{x}{s}}, \forall x \in R,$$

we have

$$\alpha g(x) = \sin \pi x\,\left[\frac{a+\sqrt{a^2+4b}}{2}\right]^{\frac{x}{s}},$$

where α is f-odd function with period s and g is a generalized Fibonacci function with period s. Hence, αg is generalized odd Fibonacci function with period s.

6. Important results

Theorem 6.1: If f is a generalized Fibonacci function with period s, then

$$\lim_{x\to\infty} \frac{f(x+ns)}{f(x+(n-1)s)} = \frac{a+\sqrt{a^2+4b}}{2} \tag{12}$$

Proof: Let

$$m = \lim_{x\to\infty} \frac{f(x+ns)}{f(x+(n-1)s)}$$

$= \lim_{x\to\infty} \frac{a\,f(x+(n-1)s)+b\,f(x+(n-2)s)}{f(x+(n-1)s)}$
$= a + b \lim_{x\to\infty} \frac{f(x+(n-2)s)}{f(x+(n-1)s)}$
$= a + b \lim_{x\to\infty} \frac{f(x+(n-1)s)}{f(x+ns)}$,

because of the property of Golden ratio of Fibonacci sequence

$$= a + b \lim_{x\to\infty} \left[\frac{f(x+ns)}{f(x+(n-1)s)}\right]^{-1}$$
$$\Rightarrow m = a + bm^{-1},$$

which gives $m = \frac{a+\sqrt{a^2+4b}}{2}$

In a similar way, we can also prove another result which can be stated as: If f is generalized odd Fibonacci function with period s, then

$$\lim_{x\to\infty} \frac{f(x+ns)}{f(x+(n-1)s)} = \frac{-a-\sqrt{a^2+4b}}{2} \quad (13)$$

7. Conclusion

Fibonacci numbers and Fibonacci functions cover a huge range of interest in advanced mathematics, Fibonacci sequence possesses wonderful and amazing properties.in this research paper properties of Generalized Fibonacci function has been discussed and these properties are also true for the particular cases, which have been constructed by various authors. These properties can also be extended to system of Generalized Tribonacci function and Generalized Tetranacci function.

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