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On some new i-convergent double sequence spaces defined by a compact operator

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ABSTRACT

In this article we introduce and study some I-convergent double sequence spaces ${}_2S^I(M)$, ${}_2S0^I(M)$, ${}_2S\infty^I(M)$ with the help of compact operator T on the real space $\mathbb{R}$ and an Orlicz function M. We study some of its topological and algebraic properties and prove some inclusion relations on these spaces.

1. Introduction

Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the sets of all natural, real, and complex numbers respectively. We denote

$$ {}_2\omega = \{x = (x_{ij}) : x_{ij} \in \mathbb{R} \text{ or } \mathbb{C}\} \tag{1.1}$$

showing the space of all real or complex double sequences.

Definition 1.1: Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator (or completely continuous linear operator), if :(i) T is linear. (ii) T maps every bounded sequence (x_k) in X onto a sequence T(x_k) in Y which has a convergent subsequence.

The set of all compact linear operators $C(\mathrm{X,Y})$ is a closed subspace of $\mathcal{B}(\mathrm{X,Y})$ and $C(\mathrm{X,Y})$ is a Banach space if Y is a Banach space. Throughout the paper, we denote ${}_2l_\infty$, ${}_2c$ and ${}_2c_0$ as the Banach spaces of bounded, convergent, and null double sequences of reals respectively with the norm:

$$\|x\| = \sup_{i,j\in\mathbb{N}} x_{ij}. \tag{1.2}$$

Following Başar and Altay (2003) and Sengonul (2007), we introduce the double sequence spaces ${}_2S$ and ${}_2S_0$ with the help of compact operator T on $\mathbb{R}$ as follows:

$$ {}_2S = \{x = (x_{ij}) \in {}_2l_\infty : T(x) \in {}_2c\}$$
$$ {}_2S_0 = \{x = (x_{ij}) \in {}_2l_\infty : T(x) \in {}_2c_0\}.$$

As a generalization of usual convergence, the concept of statistical convergent was first introduced by Fast (1951) and also independently by Buck (1953) and Schoenberg (1959) for real and complex sequences. Later on, it was further investigated from a sequence space point of view and linked with the Summability theory by Šalát (1980) and Tripathy (2004).

Definition 1.2: A double sequence $x = (x_{ij}) \in {}_2\omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, we have

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I. \tag{1.3}$$

In this case, we written $I - \lim_{ij}(x_{ij}) = L$. The notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko et al. (2000, 2005). Later on, it was studied by Šalát et al. (2004, 2005), Tripathy and Hazarika (2009, 2011), Khan and Ebadullah (2011, 2012, and 2013). Now, we recall the following definitions:

Definition 1.3: Let X be a non-empty set. Then, a family of sets $I \subseteq 2^X$ is said to be an Ideal in X if

1. $\varphi \in I$;
2. I is additive; that is, $A, B \in I \Rightarrow A \cup B \in I$;
3. I is hereditary that is, $A \in I, B \subseteq A \Rightarrow B \in I$.

An Ideal $I \subseteq 2^X$ is called non trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$.

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A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Definition 1.4: A non-empty family of sets $\mathcal{F} \subseteq 2^X$ is said to be filter on X if and only if

1. $\emptyset \notin \mathcal{F}$;
2. For, A, B $\in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$;
3. For each A $\in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.

For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I. That is,

$$\mathcal{F}(I) = \{K \subseteq N : K^c \in I\}, \text{where } K^c = N - K. \quad (1.4)$$

Definition 1.5: A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I - null if L = 0. In this case, we write

$$I - \lim x_{ij} = 0. \quad (1.5)$$

Definition 1.6: A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-Cauchy if for every $\epsilon > 0$ there exists numbers $m = m(\epsilon), n = n(\epsilon)$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I. \quad (1.6)$$

Definition 1.7: A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\} \in I. \quad (1.7)$$

Definition 1.8: A double sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies that $(\alpha_{ij}x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.9 A double sequence space E is said to be symmetric if $(x_{\pi(i,j)}) \in E$ whenever $(x_{ij}) \in E$, where $\pi(i, j)$ is a permutation on $\mathbb{N}$.

Definition 1.10: A double sequence space E is said to be sequence algebra if $(x_{ij}.y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 1.11: A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$, for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.12: Let K = $\{(n_i, k_j) : i, j \in \mathbb{N};\ n_1 < n_2 < n_3 < \ldots.$ and $k_1 < k_2 < k_3 < \ldots.\} \subseteq \mathbb{N} \times \mathbb{N}$ and E be a double sequence space. A K-step space of E is a sequence spaces:

$$\lambda_K^E = \{(\alpha_{ij}x_{ij}) : (x_{ij}) \in E\}.$$

Definition 1.13: A cannonical preimage of a sequence $(x_{n_i k_j}) \in E$ is a sequence $(b_{nk}) \in E$ defined as follows:

$$b_{n,k} = \begin{cases} x_{n,k} & \text{for } n, k \in K \\ 0, & \text{otherwise} \end{cases}$$

Definition 1.14: A sequence space E is said to be monotone if it contains the cannonical preimages of all its step spaces.

Definition.1.15: A function $M : [0, \infty) \rightarrow [0, \infty)$ is said to be an Orlicz function if it satisfies the following conditions;

1. M is continuous, convex and non-decreasing,
2. M (0) = 0, $M(x) > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Remark: (i) If the convexity of an Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called Modulus function (Tripathy and Hazarika, 2011).

(ii) If M is an Orlicz function, then $M(\lambda X) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$ (Tripathy and Hazarika, 2011). An Orlicz function M is said to satisfy Δ2-condition for all values of u if there exists constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of L > 1 (Tripathy and Hazarika, 2011).

Lindenstrauss and Tzafriri (1971) used the idea of an Orlicz function to construct the sequence space

$$l_M = \left\{x \in \omega, \sum_{k=1}^{\infty} M\left(\frac{x_k}{\rho}\right) < \infty \text{ for some } \rho > 0. \right\} \quad (1.8)$$

The space l_∞ becomes a Banach space with the norm

$$\|x\| = \inf\{\rho > 0, \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}, \quad (1.9)$$

which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(t) = t^p$ for $1 \leq p < \infty$.

Later on, some Orlicz sequence spaces were investigated by Hazarika and Esi (2013), Maddox (1970), Parashar and Choudhary (1994), Bhardwaj and Singh (2000), Et (2001), Khan et al. (2016) Tripathy and Hazarika (2011), and many others.

Initially, as a generalization of statistical convergence, the notation of I-convergence was introduced and studied by Kostyrko et al. (2000). Later on, it was studied by Khan and Ebadullah (2013), Hazarika and Esi (2013), Šalát et al. (2004, 2005) and many others. We used the following lemmas for establishing some results of this article.

Lemma 1.1: Every solid space is monotone (Tripathy and Hazarika, 2011).

Lemma 1.2: Let $F(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.

Lemma 1.3: If $I \subseteq 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.

2. Main results

In this article, we introduce the following double sequence spaces:

$${}_2S^I(M) = \left\{x = (x_{ij}) \in {}_2\omega : I - \lim M\left(\frac{|T(x_{ij}) - L|}{\rho}\right) = 0, \text{for some } L \in \mathbb{C}, \rho > 0\right\}, \quad (2.1)$$

$${}_2S_0^I(M) = \left\{x = (x_{ij}) \in {}_2\omega : I - \lim M\left(\frac{|T(x_{ij})|}{\rho}\right) = 0, \rho > 0\right\}, \quad (2.2)$$

$${}_2S_\infty^I(M) = \left\{x = (x_{ij}) \in {}_2\omega : \exists\ K > 0 s.t. \left\{i, j \in \mathbb{N} : M\left(\frac{|T(x_{ij})|}{\rho}\right) \geq K, \rho > 0\right\} \in I\right\} \quad (2.3)$$

$${}_2S_\infty(M) = \left\{x = (x_{ij}) \in {}_2\omega : \sup_{ij} M\left(\frac{|T(x_{ij})|}{\rho}\right) < \infty, \rho > 0 \in I\right\}. \quad (2.4)$$

We also denote

${}_2M_\infty^I(M) = {}_2S^I(M) \cap {}_2S_\infty(M)$ and ${}_2M_{S_0}^I(M) = {}_2S_0^I(M) \cap {}_2S_\infty(M)$.

Theorem 2.1: For any Orlicz function M, the classes of double sequence

${}_2S_0^I(M), {}_2S^I(M)$, ${}_2M_{S_0}^I(M)$ and ${}_2M_\infty^I(M)$

are linear spaces.

Proof: Let $x = (x_{ij}), (\mathcal{Y}_{ij}) \in {}_2S^I(M)$ be any two arbitrary elements and let α, β be scalars. Now, since $x = (x_{ij}), (y_{ij}) \in {}_2S^I(M) \Rightarrow \exists$ some positive numbers $L_1, L_2 \in \mathbb{C}$ and $\rho_1, \rho_2 > 0$. Such that

$$I - \lim_{ij} M\left(\frac{|T(x_{ij}) - L_1|}{\rho_1}\right) = 0, \quad (2.5)$$

and

$$I - \lim_{ij} M\left(\frac{|T(\mathcal{Y}_{ij}) - L_2|}{\rho_2}\right) = 0. \quad (2.6)$$

For any $\epsilon > o$, the sets

$$A_1 = \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{|T(x_{ij}) - L_1|}{\rho_1}\right) \geq \frac{\epsilon}{2} \in I\right\} \quad (2.7)$$

and

$$A_2 = \left\{(i, j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{|T(\mathcal{Y}_{ij}) - L_2|}{\rho_2}\right) \geq \frac{\epsilon}{2} \in I\right\}. \quad (2.8)$$

Let $\rho_3 = max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non-decreasing and convex function, we have

$$M\left(\frac{|T(\alpha x_{ij} + \beta\mathcal{Y}_{ij}) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) = M\left(\frac{|\alpha T(x_{ij}) + \beta T(\mathcal{Y}_{ij}) - \alpha L_1 - \beta L_2|}{\rho_3}\right) \leq M\left(\frac{|\alpha||T(x_{ij}) - L_1|}{\rho_3}\right) + M\left(\frac{|\beta||T(\mathcal{Y}_{ij}) - L_2|}{\rho_3}\right) < M\left(\frac{|T(x_{ij}) - L_1|}{\rho_1}\right) + M\left(\frac{|T(\mathcal{Y}_{ij}) - L_2|}{\rho_2}\right) \quad (2.9)$$

therefore, from (2.7), (2.8) and (2.9), we have

$$\left\{(i, j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{|T(\alpha x_{ij} + \beta\mathcal{Y}_{ij}) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) \geq \epsilon,\right\} \subset (A_1 \cup A_2) \in I$$

this implies that

$$\left\{(i, j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{|T(\alpha x_{ij} + \beta\mathcal{Y}_{ij}) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) \geq \epsilon,\right\} \in I$$

$$\Rightarrow \lim_{ij} M\left(\frac{|T(\alpha x_{ij} + \beta\mathcal{Y}_{ij}) - (\alpha L_1 + \beta L_2)|}{\rho_3}\right) = 0$$

$\Rightarrow\ \alpha x_{ij} + \beta y_{ij} \in 2S^I(M)$

$\Longrightarrow\ {}_2S^I(M)$ is linear space and the proof of others follow similarly.

Remark: For an Orlicz function M, the spaces ${}_2M_{s_0}^I(M)$, and ${}_2M_s^I(M)$ are Banach space normed by

$$\|x\| = inf\left\{\rho > o : \sup_{ij} M\left(\frac{|T(x_{ij})|}{\rho}\right) < 1, \rho > 0\right\}.$$

Theorem 2.2: Let $M_1; M_2$ be two Orlicz functions and satisfying Δ_2 condition, then

a) $X(M_2) \subseteq X(M_1M_2)$

b) $X(M_1) \cap X(M_2) \subseteq X(M_1 + M_2)$ for X = ${}_2S^I, {}_2S_0^I$, ${}_2M_s^I$ (M) and ${}_2M_{S_0}^I$

Proof: a) Let $x = (x_{ij}) \in {}_2S_0^I(M_2)$ be an arbitrary element. $There\ exists\ \rho > 0\ \ s.t$

$$I - \lim_{ij} M_2\left(\frac{|T(x_{ij})|}{\rho}\right) = 0. \quad (2.10)$$

Let $\epsilon > 0$ and choose $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 \leq t \leq \delta$. Put $\mathcal{Y}_{ij} = M_2\left(\frac{|T(x_{ij})|}{\rho}\right)$ and consider,

$$\lim_{ij} M_1(\mathcal{Y}_{ij}) = \lim_{\mathcal{Y}_{ij} \leq \delta, i, j \in \mathbb{N}} M_1(\mathcal{Y}_{ij}) + \lim_{\mathcal{Y}_{ij} > \delta, i, j \in \mathbb{N}} M_1(\mathcal{Y}_{ij}). \quad (2.11)$$

Now, since M_1 is an Orlicz function so we have $M_1(\lambda x) \leq \lambda M_1(x), 0 < \lambda < 1$. Therefore we have,

$$\lim_{\mathcal{Y}_{ij} \leq \delta, i, j \in \mathbb{N}} M_1(\mathcal{Y}_{ij}) \leq M_1(2) \lim_{\mathcal{Y}_{ij} \leq \delta, i, j \in \mathbb{N}} M_1(\mathcal{Y}_{ij}) \quad (2.12)$$

for $\mathcal{Y}_{ij} > \delta$, we have $\mathcal{Y}_{ij} < \frac{\mathcal{Y}_{ij}}{\delta} < 1 + \frac{\mathcal{Y}_{ij}}{\delta}$. Now, since M_1 is non-decreasing and convex, it follows that,

$$M_1(\mathcal{Y}_{ij}) < M_1\left(1 + \frac{\mathcal{Y}_{ij}}{\delta}\right) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1\left(\frac{2\mathcal{Y}_{ij}}{\delta}\right). \quad (2.13)$$

Since M_1 satisfies the Δ_2- condition we have,

$$M_1(\mathcal{Y}_{ij}) < \frac{1}{2}K\left(\frac{\mathcal{Y}_{ij}}{\delta}\right)M_1(2) + \frac{1}{2}KM_1\left(\frac{2\mathcal{Y}_{ij}}{\delta}\right). < \frac{1}{2}K\frac{\mathcal{Y}_{ij}}{\delta}M_1(2) + \frac{1}{2}K\frac{\mathcal{Y}_{ij}}{\delta}M_1(2) = K\frac{\mathcal{Y}_{ij}}{\delta}M_1(2). \quad (2.14)$$

This implies that,

$$M_1(\mathcal{Y}_{ij}) < K\left(\frac{\mathcal{Y}_{ij}}{\delta}\right)M_1(2) \quad (2.15)$$

hence, we have

$$\lim_{\mathcal{Y}_{ij} > \delta, i, j \in \mathbb{N}} M_1(\mathcal{Y}_{ij}) \leq \max\{1, k\delta^{-1} M_1(2) \lim_{\mathcal{Y}_{ij} > \delta, i, j \in \mathbb{N}} M_1(\mathcal{Y}_{ij}) \quad (2.16)$$

therefore from (2.10) and (2.11) we have

$$I-\lim_{ij} M_1(\mathcal{Y}_{ij})=0$$
$$\Rightarrow I-\lim_{ij} M_1M_2(\frac{|T(x_{ij})|}{\rho})=0.$$

This implies that $x=(x_{ij})\in {}_2S_0^I(M_1M_2)$. Hence $X(M_2)\subseteq X(M_1M_2)$ for $X={}_2S_0^I$ The other cases can be proved in similar way.

(b) Let $x=(x_{ij})\in {}_2S_0^I(M_1)\cap {}_2S_0^I(M_2)$. Let $\epsilon>0$ be given, then $\exists\,\rho>0$. Such that,

$$I-\lim_{ij} M_1(x_{ij})=0 \tag{2.17}$$

and

$$I-\lim_{ij} M_2(x_{ij})=0 \tag{2.18}$$

therefore

$$I-\lim_{ij} M_1+M_2(\frac{|T(x_{ij})|}{\rho})=I-\lim_{ij} M_1(\frac{|T(x_{ij})|}{\rho})+I-\lim_{ij} M_2(\frac{|Tx_{ij}|}{\rho})$$

from Eqs. (2.17) and (2.18)

$$\Rightarrow I-\lim_{ij}(M_1+M_2)(\frac{|T(x_{ij})|}{\rho})=0.$$

We get

$$x=(x_{ij})\in 2S_0^I(M_1+M_2).$$

Hence we get

$$x=(x_{ij})\in {}_2S_0^I(M_1+M_2).$$

For ${}_2S^I$, ${}_2M_S^I(M)$and ${}_2M_{S_0}^I$ the inclusion are similar.

Corollary: $X\subseteq X(M)$ For X $={}_2S^I,{}_2S_0^I,{}_2M_S^I$ and ${}_2M_{S_0}^I$.

Theorem 2.3: For any Orlicz function M, the spaces ${}_2S_0^I(M)$and ${}_2M_{S_0}^I(M)$ are solid and monotone.

Proof: Here we consider ${}_2S_0^I(M)$and for ${}_2M_{S_0}^I(M)$ the proof shall be similar. Let $x=x_{ij}\in {}_2S_0^I(M)$ be an arbitrary element, $\Rightarrow\exists\rho>0$ such that

$$I-\lim_{ij} M(\frac{|T(x_{ij})|}{\rho})=0. \tag{2.19}$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}|\le 1$ for $i.j\in\mathbb{N}$.

Now, M is an Orlicz function and for $\epsilon>0$, the results follows from the following inclusion

$$\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: M(\frac{|T(\alpha_{ij}x_{ij})|}{\rho})\ge\epsilon\right\}\subseteq\left\{(i,j)\in\mathbb{N}\times\mathbb{N}: M(\frac{|T(x_{ij})|}{\rho})\ge\epsilon.\right\} \tag{2.20}$$

This implies that,

$$I-\lim_{ij} M(\frac{|T(\alpha_{ij}x_{ij})|}{\rho})=0. \tag{2.21}$$

Thus we have

$$(\alpha_{ij}x_{ij})\in {}_2S_0^I(M).$$

Hence ${}_2S_0^I$ is solid. Therefore ${}_2S_0^I(M)$ is monotone. Since every solid sequence space is monotone. For ${}_2M_{S_0}^I(M)$ the proof shall be similar.

Theorem 2.4: For any Orlicz function M, the space ${}_2S^I(M)$ and ${}_2M_S^I(M)$ are neither solid nor monotone in general.

Proof: Here we give counter example for establishment of this result. Let $x={}_2S^I$and ${}_2M_S^I$. Let us consider $I=I_\delta$ and $M(x)=x^2$, for all $x=x_{ij}\in[0,\infty)$ and T an identity operator on $\mathbb{R}$. Consider, the K-step space $X_K(M)$ of $X(M)$ defined as follows: Let $x=(x_{ij})\in X(M)$ and $\mathcal{Y}=(\mathcal{Y}_{ij})\in X_K(M)$ be such that

$$(\mathcal{Y}_{ij})=\begin{cases}x_{ij}, & \text{if } i+j \text{ even,}\\ 0, & \text{otherwise.}\end{cases} \tag{2.22}$$

Consider the sequence (x_{ij}) defined by $(x_{ij})=1$ for all $i,j\in\mathbb{N}$. Then $x=(x_{ij})\in {}_2S^I(M)$ and ${}_2M_S^I(M)$, but K-step space preimage does not belong to ${}_2S^I(M)$ and ${}_2M_S^I(M)$. Thus ${}_2S^I(M)$ and ${}_2M_S^I(M)$ are not monotone and hence they are not solid.

Theorem 2.5: For an Orlicz function M and an identity operator T on $\mathbb{R}$, the spaces ${}_2S_0^I(M)$ and ${}_2S^I(M)$ are sequence algebra.

Proof: Here we consider ${}_2S_0^I(M)$. Let $x=(x_{ij})$ and $\mathcal{Y}=(\mathcal{Y}_{ij})\in {}_2S^I(M)$ be any two arbitrary elements. *There exists* $\rho_1,\rho_2>0$ such that,

$$I-\lim_{ij} M(\frac{|T(x_{ij})|}{\rho_1})=0. \tag{2.23}$$

and

$$I-\lim_{ij} M(\frac{|T(\mathcal{Y}_{ij})|}{\rho_2})=0. \tag{2.24}$$

Let $\rho=\rho_1,\rho_2>0$ then

$$M(\frac{|T(x_{ij})T(\mathcal{Y}_{ij})|}{\rho})=M(\frac{|T(x_{ij})|}{\rho_1})\,M(\frac{|T(\mathcal{Y}_{ij})|}{\rho_2})\Rightarrow I-\lim_{ij} M\left(\frac{|T(x_{ij})T(\mathcal{Y}_{ij})|}{\rho}\right)=0.$$

Therefore we have $(x_{ij}\mathcal{Y}_{ij})\in {}_2S_0^I(M)$. Hence ${}_2S_0^I(M)$ is sequence algebra.

Theorem 2.6: Let M be an Orlicz function. Then

$${}_2S_0^I(M)\subsetneq {}_2S^I(M)\subsetneq {}_2S_\infty^I(M).$$

Proof: Let M be an Orlicz function. Then, we have to show that

$${}_2S_0^I(M)\subsetneq {}_2S^I(M)\subsetneq {}_2S_\infty^I(M).$$

Firstly, ${}_2S_0^I(M) \subsetneq {}_2S^I(M)$ is obvious. Now, let $x = (x_{ij}) \in {}_2S^I(M)$ be any arbitrary element $\Rightarrow \exists\rho > 0$ such that $\Rightarrow I - \lim_{ij} M\left(\frac{|T(x_{ij}) - L|}{\rho}\right) = 0$ for $L \in \mathbb{C}$.
Now, $M\left(\frac{|T(x_{ij})|}{2\rho}\right) \leq \frac{1}{2}M\left(\frac{|T(x_{ij}) - L|}{\rho}\right) + \frac{1}{2}M\left(\frac{|L|}{\rho}\right)$.
Taking supremum over i, j to both sides, we have $x = (x_{ij}) \in {}_2S^I(M)$. Thus

$${}_2S_0^I(M) \subsetneq {}_2S^I(M) \subsetneq {}_2S_\infty^I(M).$$

Theorem 2.7: The set ${}_2M_S^I(M)$ is closed subspace of ${}_2S_\infty^I(M)$.

Proof: Let $x_{ij}^{(pq)}$ be a cauchy sequence in ${}_2M_S^I(M)$ such that $x_{ij}^{(pq)} \to x$. We show that $x = (x_{ij}) \in {}_2M_S^I(M)$.Since,$x_{ij}^{(pq)} \in {}_2M_S^I(M)$ the ther exists a_{pq}, and$\rho > 0$ such that

$$\{i, j \in \mathbb{N}: M\,(\frac{|T(x_{ij}^{(pq)})|}{\rho}) \geq \epsilon\} \in I.$$

We need to show that

1. a_{pq}, converges to a.
2. If $U = \left\{i, j \in \mathbb{N}: M\left(\frac{|T(x_{ij}) - a|}{\rho}\right) < \delta\right\}$, then $U^c \in I$.

Since $(x_{ij}^{(pq)})$ be a Cauchy sequence in ${}_2M_S^I(M)$ the for a given $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\sup_{ij} M(\frac{|T(x_{ij}^{(pq)}) - T(x_{ij}^{(rs)})|}{\rho}) < \frac{\epsilon}{3}$, for all $p, q, r, s \geq k_0$. For a given $\epsilon > 0$, we have

$$B_{pqrs} = \{i, j \in \mathbb{N}: (\frac{|T(x_{ij}^{(pq)}) - T(x_{ij}^{(rs)})|}{\rho}) < \frac{\epsilon}{3}\},$$
$$B_{pq} = \{i, j \in \mathbb{N}: (\frac{|T(x_{ij}^{(pq)}) - a_{pq}|}{\rho}) < \frac{\epsilon}{3}\},$$
$$B_{rs} = \{i, j \in \mathbb{N}: (\frac{|T(x_{ij}^{(rs)}) - a_{rs}|}{\rho}) < \frac{\epsilon}{3}\}.$$

Then $B_{pqrs}^c, B_{pq}^c, B_{rs}^c \in I$. Let $B^C = B_{pqrs}^c \cap B_{pq}^c \cap B_{rs}^c$, where $B = \{i, j \in \mathbb{N}: M(\frac{|a_{pq} - a_{rs}|}{\rho}) < \epsilon\}$, then $B^C \in I$. We choose $k_0 \in B^C$, then for each $p, q, r, s \geq k_0$ we have

$$B = \{i, j \in \mathbb{N}: M\left(\frac{|a_{pq} - a_{rs}|}{\rho}\right) < \epsilon\} \supseteq [\{i, j \in \mathbb{N}: M(\frac{|T(x_{ij}^{(pq)}) - a_{pq}|}{\rho}) < \frac{\epsilon}{3})\} \cap \{i, j \in \mathbb{N}: M(\frac{|T(x_{ij}^{(pq)}) - T(x_{ij}^{(rs)})|}{\rho} < \frac{\epsilon}{3})\} \cap \{i, j \in \mathbb{N}: M(\frac{|T(x_{ij}^{(rs)}) - a_{rs}|}{\rho}) < \frac{\epsilon}{3})\}]$$

Then (a_{pq}) is a Cauchy sequence in $\mathbb{C}$. So, there exists a scalar $a \in \mathbb{C}$ such that

$$(a_{pq}) \to a \text{ as } p, q \to \infty.$$

For the next step, let $0 < \delta < 1$ be given. Then, we show that if,

$$U = \{i, j \in \mathbb{N}: M(\frac{|T(x_{ij}) - a|}{\rho}) < \delta\}$$

then $U^c \in I$. Since $x_{ij}^{(pq)} \to x$, then there exists $p_0, q_0 \in \mathbb{N}$ such that,

$$P = \{i, j \in \mathbb{N}: M(\frac{\left|T\left(x_{ij}^{(p_0q_0)}\right) - T(x)\right|}{\rho}) < \frac{\delta}{3}\} \Rightarrow P^C \in I.$$

The numbers p_0, q_0 be so chosen such that we have

$$Q = \{i, j \in \mathbb{N}: M(\frac{|a_{p_0q_0} - a|}{\rho}) < \frac{\delta}{3}\}$$

such that $Q^C \in I$. Since $(x_{ij}^{(pq)}) \in {}_2M_S^I(M)$.
We have

$$\{i, j \in \mathbb{N}: M(\frac{\left|T\left(x_{ij}^{(p_0q_0)}\right) - a_{p_0q_0}\right|}{\rho}) \geq \delta\}$$

then we have a subset S of $\mathbb{N}$ such that $S^C \in I$, where

$$S = \{i, j \in \mathbb{N}: M(\frac{\left|T\left(x_{ij}^{(p_0q_0)}\right) - a_{p_0q_0}\right|}{\rho}) < \frac{\delta}{3}\}.$$

Let $U^C = P^C \cup Q^C \cup S^C$, where

$$U = \{i, j \in \mathbb{N}: M(\frac{|T(x) - a|}{\rho}) < \delta\}.$$

Therefore, for each $i, j \in U^C$, we have

$$\{i, j \in \mathbb{N}: M(\frac{|T(x) - a|}{\rho}) < \delta \supseteq [\{i, j \in \mathbb{N}: M(\frac{\left|T(x_{ij}^{(p_0q_0)}) - T(x)\right|}{\rho}) < \frac{\delta}{3}\} \cap \{i, j \in \mathbb{N}: M\,\frac{|a_{p_0q_0} - a|}{\rho}) < \frac{\delta}{3}\} \cap \{i, j \in \mathbb{N}: M(\frac{\left|T(x_{ij}^{(p_0q_0)}) - a_{p_0q_0}\right|}{\rho}) < \frac{\delta}{3}\}.$$

Hence the result ${}_2M_S^I(M) \subset {}_2S_\infty^I(M)$ follows.

3. Conclusion

In this paper we have studied a more general type of convergence for double sequences, that is I-Convergence as well as I-Cauchy in a more general setting i.e. compact operator is used to defined I-convergence for double sequence space. These spaces and results provide new tools to deal with the convergence problems of double sequences occurring in many branches of science and engineering.

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