International Journal of Advanced and Applied Sciences, 4(2) 2017, Pages: 35-37

Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

On the generalized C^*- valued metric spaces related with Banach fixed point theory

Özen Özer [1, *], Saleh Omran [2, 3]

[1]Department of Mathematics, Faculty of Science and Arts, Kırklareli University 39100, Kirklareli, Turkey
[2]Department of Mathematics, Faculty of Science, Taif University, Taif, Saudi Arabia
[3]Department of Mathematics, South Valley University, Quena, Egypt

A R T I C L E I N F O

Article history:
Received 24 November 2016
Received in revised form
27 January 2017
Accepted 28 January 2017

Keywords:
Fixed point theory
C^*- algebra
Positive cone

A B S T R A C T

The Banach contraction principle, which shows that every contractive mapping has a unique fixed point in a complete metric space, has been extended in many directions. One of the branches of this theory is devoted to the study of fixed points. Especially, Fixed point theory in C^*- algebra valued metric spaces has greatly developed in recent times. Also, we study on generalized C^*- algebra valued metric space and give some examples, the idea of this metric is to replace the set of real numbers by the positive cone C^*- algebras, the set of positive elements on the C^*- algebras the notation introduced recently. Also, we prove certain fixed-point theorem for a single-valued mapping in such spaces. The mapping we consider here is assumed to satisfy certain D-metric conditions with generalized fixed-point theorem. Moreover, the paper provides an application to prove the existence and uniqueness of fixed points.

1. Introduction

The study of fixed points of mappings satisfying certain contractive conditions has got big importance. The notion of D-metric space is a generalization of usual metric spaces. Dhage (1992) introduced the notion of generalized metric and claimed that D-metric convergence define a Hausdorff topology and that D -metric is sequentially continuous in all the three variables (Dhage, 1992; 1999). Then, many of authors generalized Dhages contractive proved the existence of unique fixed point of a self map in generalized metric.

El-Sayed et al. (2014) introduced the concept of quaternion metric spaces which generalizes both real and complex metric spaces and proved the fixed point theorem in normal cone metric spaces for four self-maps satisfying a general contraction condition. Huang and Zhang (2007) reviewed cone metric spaces. Rezapour and Hamlbarani (2008) obtained generalizations of the results for metric spaces and fixed point theorems of contractive mappings by providing non-normal cones and omitting the assumption of normality. Ma and Jiang (2015) established the notion of C^***-** algebra valued metric spaces, and proved some fixed point theorems for contractive and expansive mappings. Ma and Jiang (2015) introduced a concept of C^***-** algebra-valued *b* metric spaces which generalizes the concept of C^***-** algebra valued metric spaces. They also proved the existence and uniqueness results for a type of operator equation and an integral equation were given. For more details and basic definitions of the fixed point theory and C^***-** algebra we refer (Blackadar, 1986; Davidson, 1996; Gelfand and Neumark, 1943; Murphy, 1990; Pedersen, 1979). Besides, Özer and Omran (2016) have demonestrated the existence and uniqwness of the common fixed point theorem for self maps in C^***-** algebra-valued *b* metric space.

This work is motivated by the recent works on generalized metric spaces and C^***-** algebra. In this paper, we introduce the notion of generalized metric space in the C^*- algebra valued metric space. Our results can be used to investigate a large class of nonlinear problems. As an application, we discuss the existence and uniqueness for fixed point.

Definition 1.1: Let X denote a non empty set and the set off all non negative real numbers. Then X together with the function $D: X \times X \times X \to \mathbb{R}^+$ is called a D –metric space if satisfies the following properties:

(i) $D(x, y, z) = 0 \Leftrightarrow x = y = z$

* Corresponding Author.
Email Address: ozenozer39@gmail.com (Ö. Özer)
https://doi.org/10.21833/ijaas.2017.02.006
2313-626X/© 2017 The Authors. Published by IASE.

(ii) $D(x,y,z) = D(P(x,y,z))$ where, P denotes the permutation function (Symmetry).
(iii) $D(x,y,z) \le D(x,y,a) + D(x,a,z) + D(a,y,z)$ or $x,y,z,a \in X$ (Triangle inequality).

Note: Let $\mathcal{A}$ be a unital C^*- algebra with unit 1. We call $a \in \mathcal{A}$ is positive element $a \ge 0$, If $a \in \mathcal{A}$ is self-adjoint, $a^* = a$ and spectrum of a is positive real number $\sigma(a)\ \mathbb{R}^+$. The set of all positive elements in $\mathcal{A}$ denoted by $\mathcal{A}_+$.

Lemma 1.1: For any $a \in \mathcal{A}$, a^*a is positive and denote by $|a| = (a^*a)^{1/2}$. By using positive elements, one can define partial ordering $\preccurlyeq$ on $\mathcal{A}_+$ such that $x \preccurlyeq y$ if and only if $y - x \ge 0$. Besides, $\mathcal{O}$ means the zero element in $\mathcal{A}$ with the half of positive elements $\mathcal{A}_+$(Positive Cone), and also one can define a C^*-algebra valued metric space.

Definition 1.2: Let X be a non empty set. Let the mapping $d: X \times X \to \mathcal{A}$ satisfies:

(1) $0 \le d(x,y)$ for all $x,y \in X$ and $0 = d(x,y) \Leftrightarrow x = y$
(2) $d(x,y) = d(y,x)$ for all $x,y \in X$
(3) $d(x,y) \le d(x,z) + d(z,y)$ for all $x,y,z \in X$

Then d is called a C^* −algebra valued metric on X and $(X,\mathcal{A},d)$ is called C^* −algebra valued metric space.

2. Main theorem and results

In the following, we generalize the definition of the C^*- algebra valued metric space.

Definition 2.1: Let X be a non empty set and $\mathcal{A}_+$ the set of all positive elements on the C^*- algebra $\mathcal{A}$. Then X together with the function $D_{\mathcal{A}}: X \times X \times X \to \mathcal{A}_+$ is called a $D_{\mathcal{A}}$-metric space if satisfying the following conditions:

(1) $D_{\mathcal{A}}(x,y,z) = 0 \iff x = y = z$
(2) $D_{\mathcal{A}}(x,y,z) = D_{\mathcal{A}}(P(x,y,z))$, where P denotes the permutation function
(3) $D_{\mathcal{A}}(x,y,z) \preccurlyeq D_{\mathcal{A}}(x,y,a) + D_{\mathcal{A}}(x,a,z) + D_{\mathcal{A}}(a,y,z)$ for $x,y,z,a \in X$. $(X,\mathcal{A},D_{\mathcal{A}})$ is called generalized C^*- algebra valued metric space.

Definition 2.2: Let $(X,\mathcal{A},D_{\mathcal{A}})$ be a generalized C^*-algebra valued metric space. A sequence $\{x_n\} \subset X$ is asaid to be convergent in $(X,\mathcal{A},D_{\mathcal{A}})$ if $\lim_{n\to\infty} x_n = x$ and $x \in X$. It means that for any $\varepsilon > 0$ there is N such that for all $n,m > N$, $\|d(x_n,x_m,x)\| < \varepsilon$. i.e.,

$$\lim_{n,m\to\infty} D_{\mathcal{A}}(x_n,x_m,x) = 0.$$

Moreover, if for any $\varepsilon > 0$ there is N such that for all $n,m,p > N$, $\|d(x_n,x_m,x_p)\| \le \varepsilon$, then $\{x_n\}$ is called a cauchy sequence in X. We say that $(X,\mathcal{A},D_{\mathcal{A}})$ completed generalized C^*-algebra valued metric space if every cauchy sequence is convergent.

Example 2.1: If X is a Banach space, then $(X,\mathcal{A},D_{\mathcal{A}})$ is a completed generalized C^*- algebra valued metric space with respect to the metric defined by:

$$D_{\mathcal{A}}(x,y,z) = \|x-y\| + \|x-z\| + \|y-z\|$$

for, $x,y,z \in X$. The solution of the example is obvious as follow:

$$D_{\mathcal{A}}(x,y,z) = 0 \iff \|x-y\| + \|x-z\| + \|y-z\| = 0 \iff x = y = z$$

so, (1) satisfies (2) is trivial. For (3), it is clear that following equations:

$$D_{\mathcal{A}}(x,y,a) = \|x-y\| + \|x-a\| + \|y-a\|, \quad D_{\mathcal{A}}(x,a,z) = \|x-a\| + \|x-z\| + \|a-z\|$$
$$D_{\mathcal{A}}(a,y,z) = \|a-y\| + \|a-z\| + \|y-z\|$$

imply

$$D_{\mathcal{A}}(x,y,z) \preccurlyeq D_{\mathcal{A}}(x,y,a) + D_{\mathcal{A}}(a,y,z) + D_{\mathcal{A}}(x,a,z)$$

due to triangle inequality of $\|.\|$.

Definition 2.3: Let $(X,\mathcal{A},D_{\mathcal{A}})$ is a generalized C^*-algebra valued metric space. We call the mapping $T: X \to X$ generalized C^*- valued constructive mapping on X if there exist an element $a \in \mathcal{A}$ with $\|a\| < 1$ such that $d(Tx,Ty,Tz) \le a^* d(x,y,z) a$ for $x,y,z \in X$.

Theorem 2.1: If $(X,\mathcal{A},D_{\mathcal{A}})$ is a completed generalized C^*- algebra valued metric space and mapping T is a contractive mapping, then there exist a fixed point in X.

Proof: It is obvious that if $a = \mathcal{O}$. Assume that T is map from X to a single point if $a \ne \mathcal{O}$. Choose $x_0 \in X$ such that

$$\begin{aligned}
&x_{m+1} = Tx_m = T^{m+1}x_0 \text{ for } m = 1,2,\dots\\
&D_{\mathcal{A}}(x_{m+1},x_m,x_{m-1}) = D_{\mathcal{A}}(Tx_m,Tx_{m-1},Tx_{m-2})\\
&\preccurlyeq a^* D_{\mathcal{A}}(x_m,x_{m-1},x_{m-2})\, a\\
&\preccurlyeq (a^*)^2 D_{\mathcal{A}}(x_{m-1},x_m,x_{m-1})\, a^2\\
&\vdots\\
&\preccurlyeq (a^*)^{m-1} D_{\mathcal{A}}(x_2,x_1,x_0)\, a^{m-1}\\
&= (a^*)^{m-1} b\, a^{m-1}
\end{aligned}$$

Where

$D_{\mathcal{A}}(x_2,x_1,x_0) = b$ and $b \in \mathcal{A}$. So, for $n + 1 > m > p$

$$\begin{aligned}
&D_{\mathcal{A}}(x_{m+1},x_n,x_{m-1}) \preccurlyeq D_{\mathcal{A}}(x_{m+1},x_m,x_{m-1}) + D_{\mathcal{A}}(x_m,x_{m-1},x_{m-2}) + D_{\mathcal{A}}(x_{m-1},x_{m-2},x_{m-3}) + \cdots + D_{\mathcal{A}}(x_m,x_{m-1},x_{m-2}) \preccurlyeq (a^*)^{m-1} b\, a^{m-1} + \cdots + (a^*)^{m-1} b\, a^{m-1}\\
&= \sum_{k=m-1}^{n-1} (a^*)^k b a^k\\
&= \sum_{k=m-1}^{n-1} (a^*)^k b^{1/2} b^{1/2} a^k\\
&= \sum_{k=m-1}^{n-1} \left(b^{1/2} A^k\right)^* \left(b^{1/2} a^k\right)\\
&= \sum_{k=m-1}^{n-1} \left|b^{1/2} a^k\right|^2\\
&\le \left\|\sum_{k=m-1}^{n-1} \left|b^{1/2} a^k\right|^2\right\|\\
&\le \sum_{k=m-1}^{n-1} \left\|b^{1/2}\right\| \left\|a^k\right\|^2 I, \text{ (I is the unit in } \mathcal{A})\\
&= \left\|b^{1/2}\right\|^2 \sum_{k=m-1}^{n-1} \left\|a^k\right\|^2 . I\\
&\le \left\|b^{1/2}\right\|^2 \left[\frac{\|a\|^{2m-2}}{1-\|a\|}\right] . I
\end{aligned}$$

So, we can see easily,

$$\lim_{m\to\infty} \left\|b^{1/2}\right\|^2 \left[\frac{\|a\|^{2m-2}}{1-\|a\|}\right] . I = 0_{\mathcal{A}}$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $(X, \mathcal{A}, D_{\mathcal{A}})$ and by the completeness of $(X, \mathcal{A}, D_{\mathcal{A}})$ there exist an $x \in X$ such that,

$$0 \preccurlyeq D_{\mathcal{A}}(Tx, x, x) \preccurlyeq D_{\mathcal{A}}(Tx, Tx_n, x) + D_{\mathcal{A}}(Tx_n, x, x) + D_{\mathcal{A}}(x, x, Tx_n)$$

Since $\lim_{m\to\infty} x_m = x$ and $Tx_m = x_{m+1}$.

We get

$$0 \preccurlyeq D_{\mathcal{A}}(Tx, x, x) \preccurlyeq D_{\mathcal{A}}(Tx, x_{m+1}, x) + D_{\mathcal{A}}(x_{m+1}, x, x) + D_{\mathcal{A}}(x, x, x_{m+1}) < D_{\mathcal{A}}(Tx, x, x)$$

and this is a contradiction. So,

$D_{\mathcal{A}}(Tx, x, x) = 0$. Hence, $Tx = x$

(i.e. x is a fixed point of T) Also, following inequality is impossible:

$$\begin{aligned} 0 &\le \|d(x, y, z)\| = \|d(Tx, Ty, Tz)\| \le \|a^* d(x, y, z)a\| \\ &\le \|a^*\|\|d(x, y, z)\|\|a\| \\ &= \|a\|^2\|d(x, y, z)\| \\ &< \|d(x, y, z)\| \end{aligned}$$

so, $d(x, y, z) = 0$ implies that $x = y = z$. Hence we proved that the fixed point is unique.

$$\|d(x', x) + d(y', y)\| \le \|P\|. \|d(x', x) + d(y', y)\|. \|P^*\| \le \|d(x', x) + d(y', y)\|$$

this is a contradiction, and so we get:

$d(x', x) + d(y', y) = 0_{\mathcal{A}}$.
Moreover, we obtain:
$d(x', x) = 0$ and $d(y', y) = 0$

since $d(x', x)$, $d(y', y)$ are positive elements. So, we can easily see that $x' = x$ and $y' = y$ satisfy. Therefore, T has a unique Coupled Fixed Point.

3. Conclusion

Banach Contraction Principle which is one of the significant results of analysis and considered as the main source of metric fixed point theory was established in 1922. It is the most widely applied fixed point result in many branches of mathematics because it requires the structure of complete metric space with contractive condition on the map which is easy to test in this setting. The BCP has been generalized in many different directions.

It is not possible to cover all the known generalizations of the Banach Fixed Points, but some Banach Fixed Points play important role in the improvement of metric fixed point theory. In this paper, we proved the existence and the uniqueness of the Banach fixed point of the mappings satisfying a contractive condition on generalized C^*- Algebra valued metric spaces and obtained general interesting and significant result for that. We hope that the results will help the researchers in the literature of fixed point theory.

References

Blackadar B (1986). K-theory for operator algebras. Cambridge University Press, Cambridge, UK.

Davidson KR (1996). C*-algebras by example, fields institute monographs, 6. American Mathematical Society, Providence, USA.

Dhage BC (1992). Generalised metric space and mappings with fixed point. Bulletin of the Calcutta Mathematical Society, 84(4): 329–336.

Dhage BC (1999). A common fixed point principle in D-metric spaces. Bulletin of the Calcutta Mathematical Society, 91(6): 475-480.

El-Sayed AA, Omran S, and Asad AJ (2014). Fixed point theorems in quaternion-valued metric spaces. Abstract and Applied Analysis, 2014: Article ID 258985, 9 pages. https://doi.org/10.1155/2014/258985

Gelfand I and Neumark M (1943). On the imbedding of normed rings into the ring of operators in Hilbert space. Математический сборник, 12(2): 197-217.

Huang LG and Zhang X (2007). Cone metric spaces and fixed point theorems of contractive mappings. Journal of Mathematical Analysis and Applications, 332(2): 1468-1476.

Ma Z and Jiang L (2015). C* -Algebra-valued b-metric spaces and related fixed point theorems. Fixed Point Theory and Applications, 2015: 222. https://doi.org/10.1186/s13663-015-0471-6

Murphy GJ (1990). C*-algebras and operator theory. 1st Edition, Academic Press, New York, USA.

Özer Ö and Omran S (2016). Common fixed point in C*-algebra b-valued metric space. In the Applıcatıon of mathematıcs ın technıcal and natural scıences: 8th International Conference for Promoting the Application of Mathematics in Technical and Natural Sciences (AMiTaNS'16), 1773(1). 050005. AIP Publishing. https:/doi.org/10.1063/1.4964975

Pedersen GK (1979). C* - algebras and automorphism groups. Academic Press, London, UK.

Rezapour S and Hamlbarani R (2008). Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings". Journal of Mathematical Analysis and Applications, 345(2): 719-724.